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Special structures in $\mathcal{Q}(4, q)$, projective planes and its application in $L(h, k)$−colorings of their Moore Graphs

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ABSTRACT

A generalized polygon is an incidence structure that satisfies certain regularity axioms and their incidence graphs are known as Moore graphs. For any fixed non-negative integer values h and k, a $L(h, k)$-coloring is a vertex coloring in which the difference between any pair of vertices at distance one is at least h and the any pair of vertices at distance two has coloring numbers that differ by at least k. The $L(h, k)$-span is the difference between the maximum and minimum color number. The goal of this problem is to find the $L(h, k)$-coloring with the smallest span. We present three structures of the generalized $2n$-gons for $n = 3$ and $n = 4$ and use them to obtain bounds of the $L(h, k)$-number of their incidence graphs.

1. Introduction and definitions

Jacques Tits [25] introduced the generalized polygons as an incidence structure in 1959. Projective planes (generalized triangles, $n = 3$) and generalized quadrangles ($n = 4$) are special examples of generalized n-gons. Several substructures have been studied in generalized n-gons such as ovals, ovoids, spreads, t-goods, daisies, [4,8,15,16,24] among others, as those are used in many interesting related problems. In this paper we describe a structure in generalized triangles and two structures in classical generalized quadrangles and use them to obtain an $L(h, k)$-coloring of their incidence graphs.

All graphs considered in this work are finite, simple and undirected. We follow the book of Bondy and Murty [7] for terminology and notations not defined here. The *distance* between two vertices u and v is denoted by $d(u, v)$. The *diameter* is the maximum distance among any pair of vertices of the graph. The *girth* is the length of the shortest cycle of the graph.

Let $\mathcal{P}$ and $\mathcal{L}$ be disjoint non-empty sets, called the set of points and the set of lines, respectively, and let I be the point-line incidence relation. Let $\mathcal{I}$ be the ordered triple $\mathcal{I} = (\mathcal{P}, \mathcal{L}, I)$, and let $G = G[\mathcal{P}, \mathcal{L}]$ be the bipartite incidence graph on $\mathcal{P} \cup \mathcal{L}$ with edges connecting the points from $\mathcal{P}$ to their incident lines in $\mathcal{L}$. The ordered triple $\mathcal{I}$ is a symmetric generalized n-gon of order q if it satisfies the following regularity requirements.

- GP1: There exists an integer $q \geq 1$ such that every line is incident to exactly $q + 1$ points and every point is incident to exactly $q + 1$ lines.
- GP2: Any two distinct lines intersect in at most one point and there is at most one line through any two distinct points.
- GP3: The incidence graph $G = G[\mathcal{P}, \mathcal{L}]$ has diameter n and girth $2n$.

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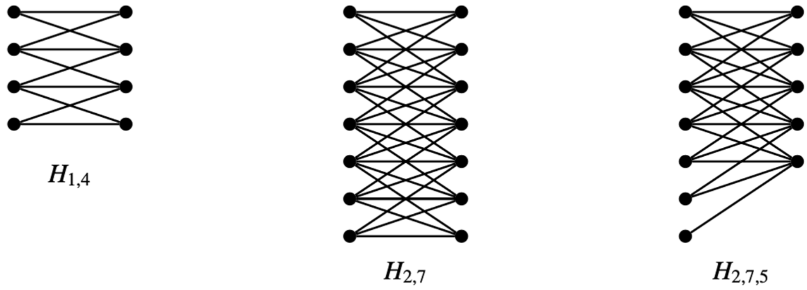

Fig. 1. Examples of $H_{s,m}$ and $H_{s,m,n}$.

The incidence graphs $G = G[\mathcal{P}, \mathcal{L}]$ have been thoroughly studied, since they are $q+1$-regular graphs with girth $2n$, and are known as $(q+1, 2n)$-Moore cages, for more information on Moore cages and generalized polygons, see [12,21,22,26]. Moore cages are interesting in many applications, for example see [1,2,5], so as an application of the structures studied and described below, we consider $L(h, k)$-colorings of the incidence graphs $G = G[\mathcal{P}, \mathcal{L}]$.

The concept of $L(h, k)$-coloring was introduced by Griggs and Yeh [18,28], in the particular case where $h = 2$ and $k = 1$, to solve problems related to the assignment frequencies or channels in a multihop radio network although it was previously mentioned by Roberts [23] in his summary on T-colorings and by Wegner [27] in the special case $h = 1$ and $k = 1$ as a problem not related to frequency assignment. The idea behind $L(h, k)$ colorings is that close locations must receive channels that differ by a certain amount k, and very close locations must receive channels that differ by a certain amount h where $h \geq k$ in order for the assignation to be efficient. An *$L(h, k)$-coloring* of a graph G is a function $\Gamma : V(G) \to \{0, 1, \ldots\}$ such that vertices at distance exactly one have colors that differ by at least h and vertices at distance exactly two have colors that differ by at least k. The span of an $L(h, k)$-coloring is the difference between the largest and the smallest color used, the minimum span over all possible $L(h, k)$ colorings of G is denoted by $\lambda_{h,k}(G)$ and is called the $L(h, k)$-number of G. The $L(h, k)$-coloring has been used to model many problems, for fixed values of h and k, for example see [3,6] and several heuristics have been studied to solve this coloring problem, for example see [10,11,14,20]. The $L(h, k)$ number has also been studied for some families of graphs for example cycles and paths [17], for more information on $L(h, k)$ colorings see [9].

In this paper we present three structures of the generalized $2n$-gons for $n = 3$ and $n = 4$ and use them to obtain bounds of the $L(h, k)$-number of their incidence graphs. In Section 2, we present an structure that appears in the complement of the incidence graph of projective planes and show bounds on $\lambda_{h,k}$ of those graphs. In Section 3, we construct a structure of points and a structure of lines, in classical generalized quadrangles, the daisy structure and the sunflower structure. Moreover, we obtain bounds for the $L(h, k)$-number their incidence graphs in terms of these structures.

2. Projective planes

Let s and n be integers such that $n \geq 2$ and $0 \leq s \leq n$. Let $H_{s,n}$ be the bipartite graph with partite sets $A = \{a_0, a_1, \ldots, a_{n-1}\}$, $B = \{b_0, b_1, \ldots, b_{n-1}\}$ and edges $\{a_i b_{i+k} \mid 0 \leq k \leq s, i+k \leq n-1\} \cup \{b_j a_{j+k} \mid 0 \leq k \leq s, j+k \leq n-1\}$. Similarly, let s, n and m be integers such that $0 \leq s \leq n \leq m$ and $2 \leq n$. Let $H_{s,m,n}$ be the bipartite graph such that if $n < m$, $H_{s,m,n} = H_{s,m} - \{b_n, \ldots, b_{m-1}\}$, and if $n = m$, $H_{s,m,n} = H_{s,m}$. Some examples of these graphs appear in Fig. 1. Throughout this paper we use the previous labeling to denote the vertices of $H_{s,m,n}$.

Let $G_1 = (X_1, Y_1)$ and $G_2 = (X_2, Y_2)$ be bipartite graphs such that $|X_1|, |X_2| \leq m$ and $|Y_1||Y_2| \leq n$. A *packing* of G_1 and G_2 into $K_{m,n}$ is a mapping $X_1 \to \{1, 2, \ldots, m\}$, $X_2 \to \{1, 2, \ldots, m\}$, and $Y_1 \to \{1, 2, \ldots, n\}$, $Y_2 \to \{1, 2, \ldots, n\}$ such that $E(G_1) \cap E(G_2) = \emptyset$.

The following lemma gives conditions on the maximum degree of a G_1 and G_2 for the existence of a packing of G_1 and G_2 into $K_{m,n}$.

Lemma 2.1 (*[19]*). *Let $G_1 = (X_1, Y_1)$ and $G_2 = (X_2, Y_2)$ be bipartite graphs such that $|X_1| = |X_2| = m$ and $|Y_1| = |Y_2| = n$. If $2\Delta(G_1)\Delta(G_2) < 1 + \max(m, n)$, then there is a packing of G_1 and G_2 into $K_{m,n}$.*

The *bipartite complement* G° of a bipartite graph $G = (A, B)$ is the graph whose vertices are $V(G)$ and $uv \in E(G^\circ)$ if $u \in A$, $v \in B$ and $uv \notin E(G)$. Observe that every subgraph H of G° and every subgraph H' of G are a packing into $K_{m,n}$ with $|A| = m$ and $|B| = n$.

Theorem 2.1. *Let q be a prime power and let G be a $(q+1, 6)$-Moore graph. Then G° has a H_{s,q^2+q+1} for every $s \leq \left\lfloor \frac{q-2}{4} \right\rfloor$.*

Proof. The $(q+1, 6)$-Moore graph is $q+1$ regular and the maximum degree of H_{s,q^2+q+1} is equal to $2s+1$. By Lemma 2.1, there is a packing of G and H_{s,q^2+q+1} into K_{q^2+q+1,q^2+q+1} whenever $2\Delta(G)\Delta(H_{s,q^2+q+1}) \leq q^2 + q + 1$. Thus, if $s \leq \left\lfloor \frac{q-2}{4} \right\rfloor$, then $2(q+1)(2s+1) < q^2 + q$ and the result follows. ■

We use Theorem 2.1 to obtain results on $\lambda_{h,k}$ of bipartite graphs, in particular, of $(q+1,6)$-Moore graphs. Recall, the *square* of a graph G, denoted by G^2, is the graph with the same vertex, in which two vertices in G^2 are adjacent when their distance in G is at most 2.

Remark 2.1. Let $G=(A,B)$ be a connected bipartite graph of girth at least 6 and let $G_2=G^2-E(G)$. Then $\lambda_{h,k}(G)\geq k(\omega(G_2)-1)$. In particular, if $diam(G)=3$ then $\lambda_{h,k}(G)\geq k(\max(|A|,|B|)-1)$.

Proof. Let $\omega(G)$ denote the maximum order of a complete subgraph of G and let H be a complete subgraph of G_2 of order $\omega(G_2)$. Since every pair of vertices in G_2 is at distance 2 in G, the color of any pair of vertices in H must differ by at least k. Thus, if $diam(G)=3$ then $V(H)=A$ or $V(H)=B$, and the result follows. ■

Theorem 2.2. *Let $G=(A,B)$ be a connected bipartite graph with partite sets A and B such that $|A|=m$ and $|B|=n$, with $m\geq n$ and girth at least 6. If G° contains $H_{1,m,n}$ as a subgraph and $h\leq 2k$, then $\lambda_{h,k}(G)\leq k(m-1)$.*

Proof. For the upper bound we exhibit an $L(h,k)$ coloring of G with $k(m-1)$ colors. By hypothesis, it follows that the bipartite complement G° contains a spanning subgraph isomorphic to $H_{1,m,n}$ with the same bipartition as G. Following the vertex labeling of $H_{1,m,n}$ over the vertices of G, let Γ be the coloring of $V(G)$ such that $\Gamma(a_i)=\Gamma(b_i)=ki$. Let $a_i,a_j\in A$ such that $d_G(a_i,a_j)=2$, then $|\Gamma(a_i)-\Gamma(a_j)|\geq k|i-j|\geq k$. Analogously, if $b_i,b_j\in B$, $|\Gamma(b_i)-\Gamma(b_j)|\geq k$. Let $a_i\in A$ and $b_j\in B$. If $|\Gamma(a_i)-\Gamma(b_j)|<h\leq 2k$, then $|i-j|\leq 1$. Hence, $a_ib_j\in E(H_{1,m,n})$ and since $H_{1,m,n}$ is a subgraph of G°, a_i and b_j are not adjacent in G. Thus, $d_G(a_i,b_j)\geq 3$, Γ is a $L(h,k)$ coloring and $\lambda_{h,k}(G)\leq k(m-1)$ and the result follows. ■

The next result is an immediate consequence of Theorem 2.2 and Remark 2.1.

Corollary 2.1. *Let $G=(A,B)$ be a connected bipartite graph with partite sets A and B such that $|A|=m$ and $|B|=n$, with $m\geq n$, girth at least 6 and $diam(G)=3$. If $h\leq 2k$, then $\lambda_{h,k}(G)=k(m-1)$.*

For instance, the incidence graphs of projective planes and the trees with diameter 3 are families of graphs that satisfy the hypothesis of Corollary 2.1. In what follows, we assume that $h>2k$.

We use the previous lemma to guarantee that $H_{s,m,n}$ is a spanning subgraph of the bipartite complement of a bipartite graph.

Theorem 2.3. *Let $G=(A,B)$ be a bipartite graph of girth at least 6 such that $\Delta<(m+1)/(4\left\lfloor\frac{h}{k}\right\rfloor+2)$ where $m=max(|A|,|B|)$. Then $\lambda_{h,k}(G)\leq k(m-1)$. Moreover, if $diam(G)=3$, then $\lambda_{h,k}(G)=k(m-1)$.*

Proof. Let $s=\left\lfloor\frac{h}{k}\right\rfloor$ and let $H_{s,m,n}$ be the bipartite graph described previously. Observe that $\Delta(H_{s,m,n})=2s+1$. By Lemma 2.1 and since $\Delta<(m+1)/(4\left\lfloor\frac{h}{k}\right\rfloor+2)$, it follows that $2(2s+1)(\Delta)<m+1$. Therefore the bipartite complement G° contains a spanning subgraph isomorphic to $H_{s,m,n}$ with the same bipartition as G.

Following the vertex labeling of $H_{s,m,n}$ over the vertices of G, let Γ be the coloring of $V(G)$ such that $\Gamma(a_i)=\Gamma(b_i)=ki$. Let $a_i,a_j\in A$, then $|\Gamma(a_i)-\Gamma(a_j)|\geq k|i-j|\geq k$. Analogously, if $b_i,b_j\in B$, then $|\Gamma(b_i)-\Gamma(b_j)|\geq k$. Let $a_i\in A$ and $b_j\in B$. If $|\Gamma(a_i)-\Gamma(b_j)|<h$, then $|i-j|\leq s$. Hence, $a_ib_j\in E(H_{s,m,n})$ and since $H_{s,m,n}$ is a subgraph of G°, a_i and b_j are not adjacent in G. Thus, $d_G(a_i,b_j)\geq 3$, Γ is a $L(h,k)$ coloring and $\lambda_{h,k}(G)\leq k(m-1)$. The result follows by the lower bound given in Remark 2.1. ■

Since $(q+1,6)$-Moore graphs are graphs with diameter 3 and girth 6, then we have the following Theorem that bounds $\lambda_{h,k}$ for every pair of h and k in terms of q or the greatest integer s such that H_{s,q^2+q+1} is a spanning subgraph of G°.

Theorem 2.4. *Let q be a prime power and let G be a $(q+1,6)$-Moore graph.*

1. *If $h\leq 2k$, then $\lambda_{h,k}(G)=k(q^2+q)$.*
2. *Let $h>2k$ and let s be the greatest integer such that H_{s,q^2+q+1} is a spanning subgraph of G°.*
   (a) *If $\left\lfloor\frac{h}{k}\right\rfloor\leq s$, then $\lambda_{h,k}(G)=k(q^2+q)$.*
   (b) *If $\left\lfloor\frac{h}{k}\right\rfloor>s$, then $k(q^2+q)\leq\lambda_{h,k}(G)\leq\left\lfloor\frac{q^2+q}{s}\right\rfloor h+\left(q^2+q \mod s\right)k$.*
   Moreover, for $q\geq 7$, $\lambda_{h,k}(G)\leq(4q+25+\frac{116}{q-5})h$. In particular, for $q\geq 121$, $\lambda_{h,k}(G)\leq(4q+26)h$.
3. *Let $h>2k$.*
   (a) *If $q>4\left\lfloor\frac{h}{k}\right\rfloor+2$, then $\lambda_{h,k}(G)=k(q^2+q)$.*
   (b) *If $q\leq 4\left\lfloor\frac{h}{k}\right\rfloor+2$, then $k(q^2+q)\leq\lambda_{h,k}(G)\leq\frac{q^2+q}{2}h$.*

Proof.

1. Let $h \le 2k$. By Corollary 2.1, with $n = m = q^2 + q + 1$, it follows that $\lambda_{h,k}(G) = k(q^2 + q)$.
2. Let $h > 2k$ and let $\alpha = min\{s, \lfloor \frac{h}{k} \rfloor\}$. For $i \in \{0, 1, \dots, q^2+q\}$, by the division algorithm, there exist c_i, d_i non-negative integers such that $i = c_i(\alpha)+d_i$, with $0 \le d_i < \alpha$. Let Γ be the coloring of $V(G)$ such that for $\Gamma(p_i) = \Gamma(l_i) = c_ih+d_ik$. Since $\alpha = min\{s, \lfloor \frac{h}{k} \rfloor\}$, H_{α,q^2+q+1} is a spanning subgraph of G°. Observe that if $x, y \in P$ or $x, y \in L$, then $d(x, y) = 2$ and $|\Gamma(x) - \Gamma(y)| \ge h > k$. Let $x \in P$ and $y \in L$. If $|\Gamma(x) - \Gamma(y)| < h$, it follows that $xy \in E(H_{\alpha,q^2+q+1})$ and by construction x and y are not adjacent in G. Hence, Γ is an $L(h, k)$ coloring of G.
   Observe that if $\lfloor \frac{h}{k} \rfloor \le s$, then $\alpha = \lfloor \frac{h}{k} \rfloor$, thus the set of colors used in Γ is
   $$\{0, k, 2k, \dots, (q^2 + q)k\}$$
   and by Remark 2.1, $\lambda_{h,k}(G) = k(q^2 + q)$.
   On the other hand, if $\lfloor \frac{h}{k} \rfloor > s$, then $\alpha = s$, and the set of colors used in Γ is
   $$\{0, k, \dots, (s-1)k, h, h+k, h+2k, \dots, h+(s-1)k, 2h, \dots, \left\lfloor \frac{q^2+q}{s} \right\rfloor h + \left(q^2+q \mod s\right) k\}$$
   and by Remark 2.1, the result follows.
   In particular, by Theorem 2.1, $\lambda_{h,k}(G) \le \left\lfloor \frac{q^2+q}{\lfloor \frac{q-2}{4} \rfloor} \right\rfloor h + \left(q^2+q \mod \lfloor \frac{q-2}{4} \rfloor\right) k$.
   Let $q \ge 7$, since $\lfloor \frac{h}{k} \rfloor > s \ge \lfloor \frac{q-2}{4} \rfloor \ge \frac{q-5}{4}$, $k < \frac{h}{\lfloor \frac{q-2}{4} \rfloor}$ and $\left(q^2+q \mod \lfloor \frac{q-2}{4} \rfloor\right) \le \lfloor \frac{q-2}{4} \rfloor - 1 = \lfloor \frac{q-6}{4} \rfloor$, it follows that $\lambda_{h,k}(G) \le (4q+25)h + \frac{116}{q-5}h$. Observe that this upper bound is quadratic on q, because $h > \lfloor \frac{q-2}{4} \rfloor k$.
3. Let $h > 2k$. If $q > 4\lfloor \frac{h}{k} \rfloor + 2$, by Theorem 2.3, with $\Delta = q + 1$, $m = q^2 + q + 1$, it follows that $\lambda_{h,k}(G) = k(q^2 + q)$. In [19] it was proved, using Lemma 2.1, and analyzing small cases, that in every $(q + 1, 6)$-Moore graph G, H_{1,q^2+q+1} is a spanning subgraph of G°. Thus, if $q \le 4\lfloor \frac{h}{k} \rfloor + 2$, and since $s \ge 1$, by item (2) and Remark 2.1, $k(q^2 + q) \le \lambda_{h,k}(G) \le \frac{q^2+q}{2}h$. ■

By Theorem 2.4, $\lambda_{2k,k}(G) = (q^2 + q)k$, which generalizes the following result presented in [19].

Theorem 2.5 (*[19]*). *If G is the incident graph of PG(2, q) then*

$$\lambda_{2,1}(G) = q^2 + q.$$

3. Classical quadrangles

We use the following coordinatization of the incidence graph of (classical) generalized quadrangles, $\mathcal{Q}(4, q)$, where $V_0 = \mathcal{P}$ and $V_1 = \mathcal{L}$.

Definition 3.1 ([2]). Let $\mathbb{F}_q$ be a finite field with $q \ge 2$ a prime power and ρ a symbol not belonging to $\mathbb{F}_q$. Let G be the incidence graph of a generalized quadrangle of order q. Let (V_0, V_1) be the bipartition of G with $V_i = \mathbb{F}_q^3 \cup \{(\rho, b, c)_i, (\rho, \rho, c)_i : b, c \in \mathbb{F}_q\} \cup \{(\rho, \rho, \rho)_i\}$, $i \in \{0, 1\}$ and edge set defined as follows:

For all $a \in \mathbb{F}_q \cup \{\rho\}$ and for all $b, c \in \mathbb{F}_q$:

$$N_G((a, b, c)_1) = \begin{cases} \{(w, aw+b, a^2w+2ab+c)_0 : w \in \mathbb{F}_q\} \cup \{(\rho, a, c)_0\} & \textit{if } a \in \mathbb{F}_q; \\ \{(c, b, w)_0 : w \in \mathbb{F}_q\} \cup \{(\rho, \rho, c)_0\} & \textit{if } a = \rho. \end{cases}$$
$N_G((\rho, \rho, c)_1) = \{(\rho, c, w)_0 : w \in \mathbb{F}_q\} \cup \{(\rho, \rho, \rho)_0\}$
$N_G((\rho, \rho, \rho)_1) = \{(\rho, \rho, w)_0 : w \in \mathbb{F}_q\} \cup \{(\rho, \rho, \rho)_0\}$.

Or equivalently, for all $i \in \mathbb{F}_q \cup \{\rho\}$ and for all $j, k \in \mathbb{F}_q$:

$$N_G((i, j, k)_0) = \begin{cases} \{(w, j-wi, w^2i-2wj+k)_1 : w \in \mathbb{F}_q\} \cup \{(\rho, j, i)_1\} & \textit{if } i \in \mathbb{F}_q; \\ \{(j, w, k)_1 : w \in \mathbb{F}_q\} \cup \{(\rho, \rho, j)_1\} & \textit{if } i = \rho. \end{cases}$$
$N_G((\rho, \rho, k)_0) = \{(\rho, w, k)_1 : w \in \mathbb{F}_q\} \cup \{(\rho, \rho, \rho)_1\}$;
$N_G((\rho, \rho, \rho)_0) = \{(\rho, \rho, w)_1 : w \in \mathbb{F}_q\} \cup \{(\rho, \rho, \rho)_1\}$.

Recall that this graph is named $(q + 1, 8)$-Moore graph, exists when q is a prime power, has order $2(q + 1)(q^2 + 1)$, is $(q + 1)$-regular, has diameter 4 and girth 8. An ovoid $\mathcal{O}$ of G is a subset of $\mathcal{P}$ of cardinality $q^2 + 1$, such that every pair of vertices in $\mathcal{O}$ is at distance four. For more information on Moore graphs or generalized quadrangles, see [12,22] respectively.

In [4] Araujo-Pardo described a special structure, the *Daisy structure*, for desarguesian projective planes. We define a similar daisy structure in the generalized quadrangle $\mathcal{Q}(4, q)$ using the coordinatization 3.1. Let G be a $(q + 1, 8)$-Moore graph with partition $(\mathcal{P}, \mathcal{L})$. For $k \in \mathbb{F}_q$, let $O_k = \{(\rho, \rho, \rho)_0\} \cup \{(i, j, k)_0 \mid i, j \in \mathbb{F}_q\}$ be a subset of $\mathcal{P}$; let $O'_k = O_k \setminus \{(\rho, \rho, \rho)_0\}$ and let $T_r = N((\rho, \rho, r)_1) \setminus \{(\rho, \rho, \rho)_0\}$ for $r \in \mathbb{F}_q \cup \{\rho\}$. The *daisy's center* is the point $(\rho, \rho, \rho)_0$, the *daisy's petals* are the sets O'_k and the *daisy's stems* are the sets T_r.

The following lemma states that the daisy structure is a partition of the points of the quadrangle.

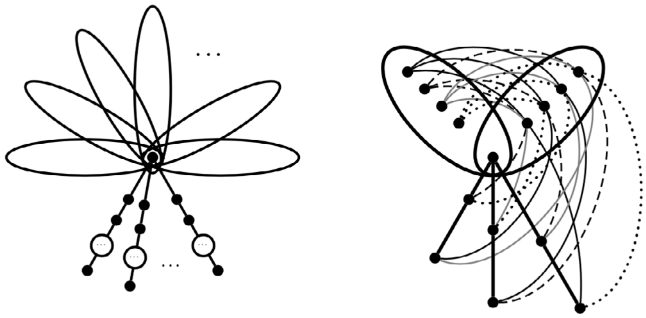

Fig. 2. At left appears the general sketch of the daisy structure of a classical generalized quadrangle. At right appears a particular example: a drawing of $GQ(2,2)$ as a daisy.

Lemma 3.1. *Let G be a $(q+1,8)$-Moore graph with bipartition $(\mathcal{P},\mathcal{L})$. The sets O_k are ovoids of G. Furthermore, the set $\mathcal{D}=\{O_0,O'_1,O'_2,\dots,O'_{q-1},T_\rho,T_0,T_1,\dots,T_{q-1}\}$ is a partition of $\mathcal{P}$.*

Proof. We use the coordinatization in Definition 3.1 to prove that the set O_k is an ovoid in G. Let $x,y\in O_k$ be two distinct vertices. Since $x,y\in\mathcal{P}$, the distance between them is even. We prove that $d(x,y)=4$. By Definition 3.1, $d((\rho,\rho,\rho)_0,(i,j,k)_0)=4$. Suppose, by contradiction, that there exist $x,y\in O_k$ such that $d(x,y)=2$. Let $x=(i,j,k)_0$, let $y=(i',j',k)_0$ and let $L=(a,b,c)_1$ be the unique vertex adjacent to x and y. Then, by Definition 3.1,

$$(i,j,k)_0=(i,ai+b,a^2i+2ab+c)_0 \qquad \text{and} \qquad (i',j',k)_0=(i',ai'+b,a^2i'+2ab+c)_0$$

Since $a^2i+2ab+c=k=a^2i'+2ab+c$, it follows that $i=i'$, $j=ai+b=ai'+b=j'$ and $x=y$, a contradiction. Hence, every pair of vertices in O_k is at distance 4.

Observe that by construction, $O_k\cap O_\ell=(\rho,\rho,\rho)_0$. Since there are no 4-cycles in G, and the distance between $(\rho,\rho,\rho)_0$ and v is 2 for every vertex $v\in T_r$, $r\in\mathbb{F}_q\cup\{\rho\}$, it follows that $T_r\cap T_s=\emptyset$. Hence, the intersection between any pair of sets in $\mathcal{D}$ is empty and their union is $\mathcal{P}$. Thus the result follows. ■

Observe that this partition, as can be seen of Fig. 2, consists in q ovoids (the petals) intersecting in one point (the center) and the lines that pass through the center (the stems).

Let $q\geq 5$ be a prime number. For the lines of the generalized quadrangle $\mathcal{Q}(4,q)$ we define a special structure, the *Sunflower structure*, using the coordinatization 3.1. Let G be a $(q+1,8)$-Moore graph with partition $(\mathcal{P},\mathcal{L})$. For $\ell\in\{0,2,\dots,q-3\}$ and $t\in\mathbb{Z}_{q+1}$ with $q=\rho$ we define the sets

$$S_{\ell,t}=\{(t,i,i-\ell)_1\mid i\in\mathbb{Z}_q\}\cup\{(t+1,j,j-\ell-1)_1\mid j\in\mathbb{Z}_q, j\neq 2^{-1}\},$$

the set $S^*_{\ell,t}=S_{\ell,t}\cup\{(\rho,\rho,t-1)_1\}$, and the set R with

$$R_t=\{(t,i,i+1)_1\mid i\in\mathbb{Z}_q\}\cup\{(t+1,2^{-1},2^{-1}-\ell-1)_1\mid \ell\in\{0,2,\dots,q-3\}\}.$$

The *sunflower's center* is the set of lines $\mathcal{C}=\{(\rho,\rho,\rho)_1\}\cup\{(\rho,\rho,i)_1\mid i\in\mathbb{Z}_q\}$, the *sunflower's petals* are the sets $S_{\ell,t}$ and the *sunflower's stems* are the sets R_t.

Lemma 3.2. *Let G be a $(q+1,8)$-Moore graph with bipartition $(\mathcal{P},\mathcal{L})$. For a fixed ℓ, every pair of vertices in $S^*_{\ell,t}$ is at distance 4 in G. Similarly, every pair of vertices in R_t is at distance 4 in G. Furthermore, the set $\mathcal{S}=\{S_{0,0},S_{0,1},\dots,S_{0,q},S_{2,0},S_{2,1},\dots,S_{2,q},\dots,S_{q-3,q},R_0,R_1,\dots,R_q,\mathcal{C}\}$ is a partition of $\mathcal{L}$*

Proof. Let $i,j\in\mathbb{Z}_q$ and $w\in\mathbb{Z}_{q+1}$ with $q=\rho$. We use the coordinatization in Definition 3.1 to prove that for fixed ℓ and t, every pair of distinct vertices $x,y\in S_{\ell,t}$ is at distance 4. Since $x,y\in\mathcal{L}$, $d(x,y)$ is even. We prove that $d(x,y)=4$. Suppose, by contradiction, that $d(x,y)=2$. We have the following three cases:

Case 1. If $x=(t,i,i-\ell)_1$ and $y=(t,i',i'-\ell)_1$, then $N(x)\cap N(y)\neq\emptyset$. Let $z\in N(x)\cap N(y)$. By Definition 3.1, $z=(w,tw+i,t^2w+2it+i-\ell)_0$ and $z=(w',tw'+i,t^2w'+2i't+i'-\ell)_0$. Then $w=w'$, and since $tw+i=tw+i'$, it follows that $i=i'$ and $x=y$, a contradiction.

Case 2. The case when $x=(t+1,j,j-\ell-1)_1$ and $y=(t+1,j',j'-\ell-1)_1$ is analogous to Case 1.

Case 3. If $x=(t,i,i-\ell)_1$ and $y=(t+1,j,j-\ell-1)_1$. Let $z\in N(x)\cap N(y)$. By Definition 3.1, $z=(w,tw+i,t^2w+2ti+i-\ell)_0$ and $z=(w',(t+1)w'+j,(t+1)^2w'+2(t+1)j+j-\ell-1)_0$. Then $w=w'$, and since $tw+i=(t+1)w+j$, it follows that $i=w+j$. Then $t^2w+2ti+i-\ell=(t+1)^2w+2(t+1)j+j-\ell-1$, if we replace $i=w+j$ it follows that $j=2^{-1}$. Thus if $j\neq 2^{-1}$, then $d(x,y)=4$ for every pair $x,y\in S_{\ell,t}$.

By Definition 3.1, if $x\in S_{\ell,t}$, then $d((\rho,\rho,t-1)_0,x)=4$, and the result follows.

Now we prove that for fixed t, every pair of distinct vertices $x, y \in R_t$ is at distance 4. Since $x, y \in \mathcal{L}$, $d(x, y)$ is even. We prove that $d(x, y) = 4$. Suppose, by contradiction, that $d(x, y) = 2$. We have the following three cases: If $x = (t, i, i+1)_1$ and $y = (t, i', i'+1)_1$, the case is analogous to Case 1 for the sets $S_{\ell,t}$.

If $x = (t+1, 2^{-1}, 2^{-1}-\ell-1)_1$ and $y = (t+1, 2^{-1}, 2^{-1}-\ell'-1)_1$, the result follows by considering the third coordinates x and y.

If $x = (t, i, i+1)_1$ and $y = (t+1, 2^{-1}, 2^{-1}-\ell-1)_1$, let $z \in N(x) \cap N(y)$. By Definition 3.1, $z = (w, tw+i, t^2w+2ti+i+1)_0$ and $z = (w', (t+1)w'+2^{-1}, (t+1)^2w'+2(t+1)2^{-1}+2^{-1}-\ell-1)_0$. Then $w = w'$, and since $tw+i = (t+1)w+2^{-1}$, it follows that $i = w+2^{-1}$. Then $t^2w+2ti+i+1 = (t+1)^2w+2(t+1)2^{-1}+2^{-1}-\ell-1$, if we replace $i = w+2^{-1}$ it follows that $\ell = -1$. That is $\ell = q-1$, which is a contradiction since $\ell \in \{0, 2, \ldots, q-3\}$. Thus $d(x, y) = 4$.

By construction, any pair of sets in $\mathcal{S}$ are mutually disjoint. So it suffices to prove that the cardinality of the union of the sets in $\mathcal{S}$ equals q^3+q^2+q+1. Observe that $|\mathcal{C}| = q+1$, $|S_{\ell,t}| = 2q-1$ and $|R_t| = q+(\frac{q-1}{2})$, and the result follows. ■

Using the daisy structure and the sunflower structure we prove the following theorem.

Theorem 3.1. *Let $q \geq 5$ be a prime number and let G be a $(q+1, 8)$-Moore graph then,*

$$(2q+2)k \leq \lambda_{h,k}(G) \leq \frac{(q+1)^2}{2}k + h$$

Proof. Let G be a $(q+1)$-Moore graph with vertex partition $(\mathcal{P}, \mathcal{L})$. For the upper bond we define the coloring Γ in terms of the daisy and the sun flower structure.

$$\Gamma(v) = \begin{cases} 0 & \text{if} \quad v \in O_0; \\ ik & \text{if} \quad v \in O_i'; \\ (q+w)k & \text{if} \quad v = (\rho, r, w)_0 \in T_r; \\ (2q-1+t)k+h & \text{if} \quad v \in R_t; \\ (3q+1+t)k+h & \text{if} \quad v \in S_{0,t}^*; \\ ((3+\frac{\ell}{2})q+1+\frac{\ell}{2}+t)k+h & \text{if} \quad v \in S_{\ell,t}; \end{cases}$$

By Lemmas 3.1 and 3.2 the coloring Γ restricted to the points or restricted to the lines is a $L(h, k)$-coloring. Since the points are colored with colors from the set $\{0, k, \ldots, (2q-1)k\}$ and the lines are colored with colors from the set $\{(2q-1)k+h, (2q)k+h, \ldots, (3q+1+\frac{q-1}{2}(q+1))k+h\}$ the colors of adjacent vertices are at distance at least h.

Hence, Γ uses $\frac{(q+1)^2}{2}k+h$ colors and the upper bound follows.

For the lower bound, assume for contradiction, that there exists an $L_{h,k}$ coloring Γ that uses at most $(2q+2)k-1$ colors. Let $A_j = \{jk, jk+1, \ldots, (j+1)k-1\}$ for $0 \leq j \leq 2q+1$ be a partition of the numbers used in Γ. Observe that $\Gamma^{-1}(A_j)$ is a set of vertices which are pairwise at distance 3 or 4. In the proof of Theorem 3.2 of [13] the authors proved that every subset of vertices of a $(q+1, 8)$-Moore graph which contains vertices at distance 3 or 4 pairwise contains at most q^2+1 vertices. By the pigeonhole principle, there exists a r such that A_r contains at least q^2+2 vertices, a contradiction. Hence, Γ must use at least $(2q+2)k$ colors and the result follows. ■

We believe the study of these structures can help to obtain bounds on other distance-based graph parameters on the incidence graphs of projective planes and classical quadrangles.

Declaration of competing interest

The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

Data availability

No data was used for the research described in the article.

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References

[1] D. Abrego, S. Fernández-Merchant, M. Neubauer, W. Watkins, D-Optimal weighing designs for $n \equiv -1 \pmod 4$ objects and a large number of weighings, Linear Algebra Appl. 374 (2003) 175–218.
[2] M. Abreu, G. Araujo, C. Balbuena, D. Labbate, A construction of small $(q-1)$-regular graphs of girth 8, Electron. J. Combin. 22 (2) (2015) #P2.10.
[3] K.A. Aly, P.W. Dowd, A class of scalable optical interconnection networks through discrete broadcast-select multi-domain WDM, in: Proc. IEEE INFOCOM, Toronto, Ontario, Canada, 1994.
[4] M.G. Araujo-Pardo, Daisy structure in desarguesian projective planes, J. Aust. Math. Soc. 74 (2) (2003) 145–153.
[5] M. Atici, R. Crawford, C. Ernst, The integrity of small cage graphs, Australas. J. Combin. 43 (2009) 39–55.
[6] A.A. Bertossi, M.A. Bonuccelli, Code assignment for hidden terminal interference avoidance in multihop packet radio networks, IEEE/ACM Trans. Netw. 3 (4) (1995) 441–449.
[7] J.A. Bondy, U.S.R. Murty, Graph Theory, GTM 244, Springer, Berlin, 2008.
[8] F. Buekenhout (Ed.), The Handbook of Incidence Geometry, Elsevier Science Publishers B.V., Amsterdam, The Netherlands, 1995.
[9] T. Calamoneri, The $L(h, k)$-labelling problem: A survey and annotated bibliography, Comput. J. 49–5 (2006) 585–608.
[10] D. Castelino, S. Hurley, N.M. Stephens, A tabu search algorithm for frequency assignment, Ann. Oper. Res. 63 (1996) 301–319.
[11] M. Duque-Anton, D. Kunz, B. Rüber, Channel assignment for cellular radio using simulated annealing, IEEE Trans. Veh. Technol. 42 (1993) 14–21.
[12] G. Exoo, R. Jajcay, Dynamic cage survey, Electron. J. Combin. (2013) #DS16.
[13] J. Fresán-Figueroa, D. González-Moreno, M. Olsen, On the packing chromatic number of Moore graphs, Discrete Appl. Math. 289 (2021) 185–193.
[14] N. Funabiki, Y. Takefuji, A neural network parallel algorithm for channel assignment problem in cellular radio networks, IEEE Trans. Veh. Technol. 41 (1992) 430–437.
[15] M. Funk, M. Labbate, V. Napolitano, Tactical (de-)compositions of symmetric configurations, Discrete Math. 309 (4) (2009) 741–747.
[16] A. Gács, T. Héger, On geometric constructions of (k, g)-graphs, Contrib. Discrete Math. 3 (1) (2008) 63–80.
[17] J.P. Georges, D.W. Mauro, Generalized vertex labeling with a condition at distance two, Congr. Numer. 109 (1995) 141–159.
[18] J.R. Griggs, R.K. Yeh, Labeling graphs with a condition at distance 2, SIAM J. Discrete Math. 5 (1992) 586–595.
[19] J.H. Kang, Coloring of Metric Spaces and Labeling of Graphs (Ph.D. thesis), University of Illinois at Urbana-Champaign, 2004.
[20] W.K. Lai, G.G. Coghill, Channel assignment through evolutionary optimization, IEEE Trans. Veh. Technol. 45 (1996) 91–96.
[21] M. Miller, J. Sirán, Moore graphs and beyond: A survey of the degree/diameter problem, Electron. J. Combin. (2013) #DS14.
[22] S.E. Payne, J.A. Thas, Finite Generalized Quadrangles. Vol. 9, European Mathematical Society, 2009.
[23] F.S. Roberts, T-colorings of graphs: recent results and open problems, Discrete Math. 93 (1991) 229–245.
[24] J.A. Thas, S.E. Payne, Spreads and ovoids in finite generalized quadrangles, Geom. Dedicata 52 (3) (1994) 227–253.
[25] J. Tits, Sur la trialité et certains groupes qui s'en d'eduisent, Inst. Hautes Eétudes Sci. Publ. Math. 2 (1959) 13–60.
[26] H. Van Maldeghem, Generalized Polygons, Springer Science & Business Media, 2012.
[27] G. Wegner, Graphs with Given Diameter and a Coloring Problem, Technical Report, University of Dortmund, Dortmund, 1997.
[28] R.K. Yeh, Labeling Graphs with a Condition at Distance Two (Ph.D. thesis), University of South Carolina, Columbia, South Carolina, 1990.