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On (r, g, χ)- graphs and cages for infinitely many values of regularity r, girth g and chromatic number χ .

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Abstract

For integers $r \geq 2$, $g \geq 3$ and $\chi \geq 2$, an (r, g, χ)-graph is an r-regular graph with girth g and chromatic number χ. Such a graph of minimum order is called an (r, g, χ)-cage. Here we prove the existence of (r, g, χ)-graphs for all r and even g when $\chi = 2$ and for all r and g when $\chi = 3$. Furthermore, using both existence proofs and explicit constructions we give examples of (r, g, χ)-graphs for infinitely many values of r, g, χ.

Keywords: Graphs, cages, girth, chromatic number

Math. Subj. Class.: 05C10, 05C15

1 Introduction

In this paper, we consider only simple and finite graphs. We explore a generalization of the *Cage Problem*. An (r, g)-*graph* is an r-regular graph with girth g. An (r, g)-*cage* is an (r, g)-graph with the minimum possible number of vertices among all (r, g)-graphs, and the order of an (r, g)-cage is denoted by $n(r, g)$. The Cage Problem involves finding (r, g)-cages, and it is well-known that (r, g)-cages have been determined only for very limited sets of parameter pairs (r, g). Moore's lower bound $n_0(r, g)$, as outlined by Moore (see [7]), is a simple lower bound for $n(r, g)$. Cages that achieve this bound are referred to as *Moore cages*. Many generalizations of this concept have been studied; for example, birregular cages, bipartite birregular cages, mixed cages, Cayley cages and vertex transitive cages.

Here, we focus on a generalization proposed by Araujo, Berikkyzy and Lesniak in [1]. The authors introduced the notions of (r, g, χ)-graphs and (r, g, χ)-cages. For integers $r \geq 2$, $g \geq 3$ and $\chi \geq 2$, an (r, g, χ)-*graph* is an r-regular graph with girth g and chromatic number χ, and an (r, g, χ)-*cage* is such a graph of minimum order. We use $n(r, g, \chi)$ to denote the order an (r, g, χ)-cage. Extending the values of r, g, χ, for which (r, g, χ)-graphs and cages are known to exist will be our primary goal here. We will also bound the values of $n(r, g, \chi)$ in some cases.

Previous papers considered any 2 of the parameters r, g, χ. For example, the classic definition of (r, g)-graph is an r-regular graph with girth g. We will denote such a graph as an $(r, g, -)$-graph. Similarly, in [12] the author defines an $(r; \chi)$-graph to be an r-regular graph with chromatic number χ. We will denote such a graph as an $(r, -, \chi)$-graph.
Finally, Erdös [9] considered graphs with a given chromatic number $\chi > 2$ and girth greater than $g \geq 3$. Adding a disjoint cycle of length g to such a graph gives us a graph with girth g and chromatic number χ, which we call a $(-, g, \chi)$-graph.

In [1], the authors proved the existence of (r, g, χ)-cages for any $r \geq 2$, girth $g \geq 3$ and $\chi = 3$. Also, they characterized the $(r, 3, 3)$-cages for any $r \geq 2$. Finally, they studied balanced (r, g, χ)-graphs, that is (r, g, χ)-graphs for which there is a χ-coloring where the color classes differ by at most 1.

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It is important to note that in [1] the authors assumed the Folklore conjecture (see also [11]) which states that, for even values of girth g, all (r, g)-cages are bipartite or equivalently that they have chromatic number equal to 2, to prove the existence of $(r, g, 3)$-cages for any r and any g. In this paper we prove, constructively, the existence of $(r, g, 2)$-cages for all $r \geq 2$ and even girth $g \geq 4$. Consequently, the existence of (r, g, χ)-cages for any $r \geq 2$, and girth $g \geq 3$ and $\chi = 3$ given in [1] follows immediately.

The authors of [1] did not explore the existence of (r, g, χ)-graphs for $4 \leq \chi \leq r + 1$. In this paper we also illustrate the existence/construction of (r, g, χ)-graphs for infinitely many values of r, g, χ.

2 The existence of (r, g, χ)-graphs for given g and χ and infinitely many r.

It is well known that for any positive integer k, there exists a triangle-free k-chromatic graph (see [4] for a proof of this result given by Mycielski also in [10]). In fact, the graphs constructed in this proof have girth equal to 4. Moreover, one of the nicest applications of the probabilistic method on graph theory was given by Erdös in 1959 (see [9]). He proved that, given integers $g \geq 3$ and $\chi \geq 2$ there exists a graph with girth g and chromatic number χ.

In addition to this result, to prove the main theorem of this section, we need to define the following family of trees. Let $g \geq 3$ and $r \geq 3$ be integers. We define the *semi-complete Moore tree* $\mathcal{T}_g^r$ as the tree obtained from the complete r-ary tree with height g by removing two branches from the rooted vertex (see Figure 1). Observe that

$$|V(\mathcal{T}_g^r)| = 1 + (r-2) + \sum_{i=1}^{g-1} (r-2)(r-1)^i,$$

which is a multiple of $r - 1$.

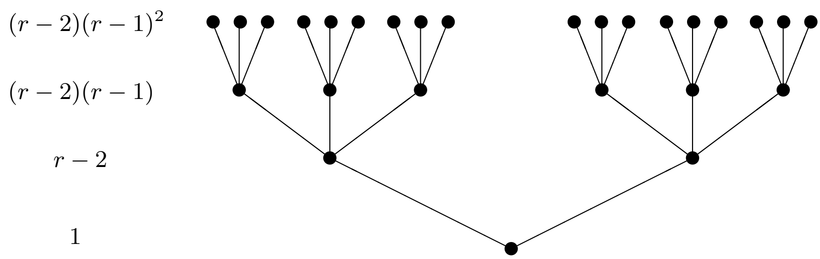

Figure 1: The semi Moore tree $\mathcal{T}_3^4$.

Theorem 2.1. *Let $g \geq 3$ and $\chi \geq 2$ be integers. Then there is an integer r^* such that for every $r > r^*$ there exists a (r, g, χ)-graph.*

Proof. Let F be a graph with girth g, chromatic number χ and the the minimum possible order, and let $|V(F)| = n_1$. Let $\Delta(F) = r^* \geq 2$ and let $r \in \mathbb{Z}$, such that $r > r^*$.

Define the set S as the collection of all graphs of order

$$n = (r-1)n_1 + 2(r-1) + \sum_{i=1}^{g-1}(r-2)(r-1)^i + \sum_{i=1}^{g}(r-2)(r-1)^i,$$

with girth g, maximum degree r, chromatic number χ that satisfy the following property: if x is a vertex of a graph $G \in S$ such that $d(x) = r$, then there is an edge e incident with x such that $\chi(G-e) = \chi$ and $g(G-e) = g$.

We first show that $S \neq \emptyset$. Let H be the graph constructed by considering the union of $r-1$ copies of F and the semi Moore trees $\mathcal{T}_g^r$ and a $\mathcal{T}_{g+1}^r$. Note that $|V(H)| = (r-1)|V(F)| + |V(\mathcal{T}_g^r) + |V(\mathcal{T}_{g+1}^r)| = n$. If $x \in V(H)$ and $d(x) = r$, then $x \in V(\mathcal{T}_g^r) \cup V(\mathcal{T}_{g+1}^r)$, and if e is an edge incident with x, then $\chi(G-e) = \chi$ and $g(G-e) = g$. Therefore $G \in S$, and the claim follows.

For every graph G in S let

$$M(G) = \{x \in V(G) \mid d(x) < r\},$$

and let $m(G) = \max\{d(x,y) : x, y \in M(G)\}$.

Let S' be the set of graphs in S with the maximum number of edges, and let S'' be the graphs G in S' for which $|M(G)|$ is maximum. Let G be a graph in S'' such that $m(G)$ is maximum and let $u, v \in M(G)$ such that $d(u,v) = m(G)$. The proof will be complete if we show that $M(G) = \emptyset$.

Claim 2.2. $m(G) \leq g-1$. Suppose that $m(G) \geq g$. Let $P = (u, x_1, x_2, \ldots, x_{g-1}, x_g = v)$ be an uv-geodesic. If there exists a χ-coloring Γ of $V(G)$ such that $\Gamma(u) \neq \Gamma(v)$, then the graph G' obtained from G by adding the edge uv is in S and $|E(G')| > |E(G)|$, contradicting that the size of G is the largest possible. So, for every χ-coloring Γ of $V(G)$ it follows that $\Gamma(u) = \Gamma(v)$. Since $x_{g-1}v \in E(G)$, we have that $\Gamma(u) \neq \Gamma(x_{g-1})$. Hence the graph G' obtained from G by adding the edge ux_{g-1} is also in S and $|E(G')| > |E(G)|$, contradicting the selection of G. Therefore $d(u,v) = m(G) \leq g-1$.

Let W be the set of vertices w in G such that either $d(u,w) \leq g-1$ or $d(v,w) \leq g$. By counting the set of vertices at distance at most $g-1$ from u and at distance at most g from v it follows that

$$|W| \leq \sum_{i=1}^{g-1}(r-1)^i + \sum_{i=1}^{g}(r-1)^i < |V(G)|.$$

Since $|V(G)| > |W|$, there exists a vertex x_1 in $V(G) \setminus W$ and $d(u,x_1) \geq g$ and $d(v,x_1) \geq g+1$. As $d(u,x_1) > m(G)$ and $u \in M(G)$, then $x_1 \notin M(G)$ and so $d(x_1) = r$. Moreover, there exists an edge e incident with x_1 whose deletion does not affect the chromatic number and the girth of G (because $G \in S$). Assume that $e = x_1x_2$.

Observe that $d(v,x_1) \leq d(v,x_2) + 1$, therefore $d(v,x_2) \geq d(v,x_1) - 1 \geq g+1-1 = g > m(G)$, and $x_2 \notin M(G)$, so $d(x_2) = r$.

If there exists a χ-coloring Γ of $V(G)$ such that $\Gamma(x_1) \neq \Gamma(u)$, then let G' be the graph obtained from G by deleting the edge x_1x_2 and adding the edge ux_1. Note that $G' \in S'$, $x_2 \in M(G')$ and every vertex x in $M(G)$ (with the possible exception of u) belongs to $M(G')$. By the way was chosen G, it follows that $|M(G')| \leq |M(G)|$, so $u \notin M(G')$ and $|M(G')| = |M(G)|$. Therefor, $d_{G'}(u) = r$ and $G' \in S''$. If $|M(G')| = 1$, then

$M(G) = \{u\}$ and $u = v$. Since $d_{G'}(u) = r$, hence, $d_G(u) = r - 1$. Therefore G contains exactly one vertex of degree $r - 1$ and $n - 1$ vertices of degree r. Then $r - 1$ and $n - 1$ are even, yielding a contradiction because n is a multiple of $r - 1$.

Therefore $v, x_2 \in M(G')$ and then

$$d_{G'}(v, w_2) \leq m(G') \leq m(G) \leq g - 1.$$

Let T be an vx_2-geodesic in G'. If T is totally contained in G, then the length of T is at least g, giving a contradiction. Therefore T uses the edge ux_1 and either T contains an vu-path totally contained in G of length at least $m(G)$ or an vx_1-path, also contained in G, of length at least $g + 1$. In both cases $l(T) > m(G')$, a contradiction.

Thus, every χ-coloring of $V(G)$ assigns the same color to u and x_1. Let Γ be a χ-coloring of $V(G)$. Since $x_1x_2 \in E(G)$, it follows that $\Gamma(u) \neq \Gamma(x_2)$. Let G' be the graph obtained from G by deleting the edge x_1x_2 and adding the edge ux_2. Note that $v, x_1 \in M(G')$. Let T be an vx_1-geodesic in G'. If T is totally contained in G, then the length of T is at least $g + 1$, leading to a contradiction. Therefore T uses the edge ux_2 and either T contains an vu-path totally contained in G of length at least $m(G)$ or an vx_1-path, also contained in G of length at least $g + 1$. In both cases $l(T) > m(G')$, a contradiction. Therefore $M(G) = \emptyset$. □

If we consider the order of the graph used in the previous proof, we can obtain an upper bound for the order of an (r, g, χ)-cage (in terms of the order of a $(-, g, \chi)$-cage) is obtained. For more references on the order of graphs with girth g and chromatic number χ, please consult [6].

Corollary 2.3. *Let $g \geq 3$ and $\chi \geq 2$ be integers. Let G be an $(-, g, \chi)$-graph with order n_1. If $r > \Delta(G)$, then*

$$n(r, g, \chi) \leq r(r - 1)^g + n_1(r - 1).$$

Proof. The result is obtained by simplifying the expression $(r-1)n_1+2(r-1)+\sum_{i=1}^{g-1}(r-2)(r-1)^i + \sum_{i=1}^{g}(r-2)(r-1)^i$. □

The Brinkmann graph has order 21 and is an example of a $(4, 5, 4)$-cage [3]. Chvátal [5] showed that the Grötzsch graph, which has maximum degree 4, is the smallest 4-chromatic graph with girth 4. Using these results and Corollary 2.3, the following conclusions are obtained.

Corollary 2.4. *If $r \geq 5$ is an integer, then*

- $n(r, 5, 4) \leq (r - 1)(r^5 - 4r^4 + 6r^3 - 4r^2 + r + 21)$.
- $n(r, 4, 4) \leq (r - 1)(r^4 - 3r^3 + 3r^2 - r + 11)$.

3 Constructive proofs

In this section we give constructive proofs of the existence of (r, g, χ)-graphs for various values of these parameters r, g, χ.

Certainly, $(r, g, 2)$-graphs are only possible for even g since a graph has chromatic number 2 if and only if it is bipartite, and a graph is bipartite if and only if it has no odd cycles.

Let G and H be two graphs. The *Kronecker product* of G and H, denoted as $G \otimes H$, is the graph with vertex set $V(G \otimes H) = V(G) \times V(H)$ and edge set $E(G \otimes H) = \{(u, v)(u', v') \mid uu' \in E(G) \text{ and } vv' \in E(H)\}$. Observe that $G \otimes H$ has order $|V(G)| \cdot |V(H)|$ and size $2|E(G)| \cdot |E(H)|$. Moreover, if $(u, v) \in V(G \otimes H)$, then $d_{G \otimes H}(u, v) = d_G(u) \cdot d_H(v)$.

It is known that if G is a graph with girth g, then $G \otimes K_2$ is a bipartite graph, and it is disconnected if and only if G is bipartite. Moreover, if g is even, then the girth of $G \otimes K_2$ is equal to g [14].

We use the Kronecker product to show that $(r, g, 2)$-graphs exist for all $r \geq 2$ and even $g \geq 4$.

Theorem 3.1. *For every pair of integers $r \geq 2$ and $g \geq 4$ even, there exists an $(r, g, 2)$-graph.*

Proof. Let $r \geq 2$, $g \geq 4$ be even and let H be an (r, g)-graph. If H is bipartite, then H is an $(r, g, 2)$-graph. If H is not a bipartite graph, then consider the graph $G = H \otimes K_2$. By properties of Kronecker product G is a bipartite (r, g)-graph and thus, $\chi(G) = 2$ and the result follows. □

As noted earlier, this result implies, from [1], that $(r, g, 3)$- graphs exist for all $r \geq 2$ and $g \geq 3$.

In [1], the order of $(r, 3, 3)$-cages were determined for all r. The next value of interest is the order of $(r, 4, 3)$-cages. Next, we bound this number.

Theorem 3.2. *Let $r \geq 2$, then:*

$$3r/2 \leq n(r, 4, 3) \leq 4r - 2.$$

Proof. It is quite easy to show that every $(r, g, 3)$-graph has order at least $3r/2$. For the upper bound, consider the complete bipartite graph $K_{r,r}$, and let M be a matching of $K_{r,r}$. Let $w_1, w_2, \ldots, w_r$ be a set of vertices obtained by subdividing the edges of M. Finally, let G be the graph obtained by adding $r - 2$ vertices $u_1, u_2, \ldots, u_{r-2}$ and making them adjacent to each vertex in $w_1, w_2, \ldots, w_r$. Observe that G is a $(r, 4, 3)$-graph of order $4r - 2$. Thus the order of a $(r, 4, 3)$-cage is bounded between $3r/2$ and $4r - 2$. □

In Theorem 3.1 we showed that for given girth $g \geq 3$ and chromatic number $\chi \geq 2$, we can find (r, g, χ)-graphs for r sufficiently large. In our next theorem we show that beginning with a graph of girth $g \in \{4, 5, 6\}$, chromatic number $\chi \geq 2$ and maximum degree Δ, there is a (r, g, χ)-graph with $r = \Delta$.

Theorem 3.3. *Let G be a $(-, g, \chi)$-graph for $g \in \{4, 5, 6\}$ with maximum degree Δ. Then there exists a (Δ, g, χ)-graph.*

Proof. For any graph G let $Z(G) = \{v \in V(G) \mid d(v) < \Delta\}$ and let $D(G) = \{k \mid k = d(v)$ for some $v \in Z(G)\}$, that is, $D(G)$ is the set of degrees of vertices in G less than Δ. For $g = 4$, let G be a graph with girth 4 and chromatic number χ. Such a graph exists by the aforementioned results of Mycielski [10] and Erdös [9]. Let G' be a copy of G, and let

v' denote the copy of the vertex $v \in V(G)$ in G'. Consider the graph $G_1 = G \cup G' + E'$, where $E' = \{vv' \mid v \in Z(G)\}$. Note that E' forms a matching between the sets $V(G)$ and $V(G')$, ensuring that the girth of G_1 remains 4. Moreover, G_1 is also χ-chromatic. Let $\Gamma : V(G) \to \{1, 2, \ldots, \chi\} = [\chi]$ be a proper vertex coloring of G and let σ be a permutation of $[\chi]$ without fixed points, that is, $\sigma(i) \neq i$ for every $i \in [\chi]$. Let φ be the vertex coloring of G_i such that $\varphi(v) = \Gamma(v)$ and $\varphi(v') = \sigma(\Gamma(v))$. Then φ is a proper χ-coloring of G_1, and since G_1 contains a copy of G then it must be χ-chromatic. Finally, if $Z \neq \emptyset$, then $D(G_1) = (D(G) + 1) \setminus \{\Delta\} = \{a + 1 \mid a \in D(G)\} \setminus \{\Delta\}$.

Next, let G'_1 be a copy of G_1 and form a graph $G_2 = G_1 \cup G'_1 + E'_1$, where $E'_1 = \{uu' \mid u \in Z(G_1)\}$. Using the same reasoning as the previous case it follows that $\chi(G_2) = \chi$, and observe that $D(G_2) = (D(G_1) + 1) \setminus \{\Delta\}$.

We can continue this process by considering a graph G_i, creating a copy of it, and adding a matching between the vertices in $Z(G_i)$.

This process ends after $\Delta(G) - \delta(G)$ steps. The resulting graph is a Δ-regular graph with girth 4 and chromatic number χ.

For $g \in \{5, 6\}$, let G be a graph with girth g, chromatic number χ and maximum degree Δ.

Let (M, S) be such that M is a maximum matching in the subgraph of G induced by $Z(G)$ and $S = Z(G) \setminus V(M)$. Then S is an independent set, since otherwise we can add an edge to M, contradicting its maximality.

We can construct a graph G_1 such that $V(G_1) = V(G) \times \{1, 2\}$ and $E(G_1) = C_1 \cup C_2 \cup L_s$ where $C_i = \{\{(u, i), (v, i)\} \mid \{u, v\} \in E(G)\}$ for $i = 1, 2$ and $L_s = \{\{(u, 1), (u, 2)\} \mid u \in S\}$. Observe that G_1 was constructed as two copies of G and and a matching between vertices in S. Hence, G_1 has girth g since the cycles that go through an edge in L_s must have length at least 6, because S is an independent set. Additionally, G_1 is χ-chromatic since G_1 contains a copy of G and we can give a χ-coloring as follows: Let Γ be a χ-coloring of G and let σ be a permutation of the set $\{1, 2, \ldots, \chi\}$ with no fixed points, that is $\sigma(i) \neq i$ for any i. Then we can color G_1 with the coloring ρ such that $\rho(u, 1) = \Gamma(u)$ and $\rho(u, 2) = \sigma(\Gamma(u))$. By construction ρ is a proper χ-coloring, therefore G_1 is χ-chromatic.

Now we construct a graph G_2 such that $V(G_2) = V(G_1) \times \{1, 2\}$, and $E(G_2) = E_1 \cup E_2 \cup L_M$, where $E_k = \{\{(u, i, k), (v, j, k)\} \mid \{(u, i), (v, j)\} \in E(G_1)\}$ for $k = 1, 2$ and let M be the perfect matching in $Z(G) \setminus S$. Define $L_M = \bigcup_{\{a,b\} \in M} L_{\{a,b\}}$ and $L_{\{a,b\}} = \{\{(a, 1, 1), (a, 2, 1)\}, \{(b, 2, 1), (a, 1, 2)\}, \{(b, 1, 2), (a, 2, 2)\}, \{(b, 2, 2), (b, 1, 1)\}\}$. Finally, G_2 is constructed as two copies of G_1 and the edges in L_m joining them. Observe that G_2 has girth g since M is a matching and the new cycles formed with the edges in L_M are of order at least 6. Similarly, G_2 is χ-chromatic since it contains a copy of G_1 and we can color G_2 as follows. Let ρ be the χ-coloring of G_1 previously defined. Let φ be such that $\varphi((a, i, j)) = \rho((a, i))$. Then φ is a proper χ-coloring of G_2 since the edges in the copies of G_1 and the edges in L_M are well colored. Moreover observe that $D(G_2) = \{a + 1 \mid a \in D(G)\} \setminus \{\Delta\}$. So the graph G_2 is a graph such that the degree of the vertices of G that do not have degree Δ is increased by one. By repeating this construction we can obtain a graph Δ-regular, with girth g and chromatic number χ and the result follows.

□

To obtain a $(r + 1, g, k)$-graph from a (r, g, k)-graph, we use a restatement of Sachs' classic construction of (r, g)-graphs in [13]. Exoo and Jajcay introduced this adaptation in

[7] and called it a *generalized truncation*. A generalized truncation requires four elements.

1. A graph of order r with vertex set $V(H) = \{u_1, u_2, \ldots, u_r\}$.
2. A finite r-regular graph G.
3. A set of darts $D(G) = \{\overrightarrow{vw} \mid v, w \in V(G)\}$. Then, there is a pair of opposing *darts* associated with each edge of G, such that each pair of darts connects the two endpoints of its respective edge, but each dart starts at a different end. Thus, $|D(G)| = 2|E(G)|$.
4. A function mapping ρ from $D(G)$ to the set $\{1, 2, \ldots, r\}$ satisfying that for any vertex v of G, the darts starting from v all receive different labels, i.e. ρ maps the darts emanating from each vertex onto the set $\{1, 2, \ldots, r\}$. Each ρ mapping is called a *vertex-neighborhood labeling* of G. Because there is no restriction to the values of ρ for the darts ending in each vertex, any assignation of all the values $\{1, 2, \ldots, r\}$ to the darts starting in each vertex can be a vertex-neighborhood labeling of G.

The *generalized truncation* of a k-regular graph G with a vertex-neighborhood labeling ρ by the graph H is the graph $T(G, \rho, H)$ such that:

- $V(T(G, \rho, H)) = \{u_i^v \mid i \in \{1, 2, \ldots, r\}$ and $v \in V(G)\}$ and
- $E(T(G, \rho, H)) = \{u_i^v u_j^v \mid (u_i, u_j) \in E(G)$ and $v \in V(G)\} \cup \{u_i^v u_j^w \mid v, w \in V(G), v \neq w, vw \in E(G), \rho(\overrightarrow{vw}) = j,$ and $\rho(\overrightarrow{wv})) = i\}$.

Two known results that help understand the effects of this construction and its relationship with the girths of G and H are the following.

Lemma 3.4 ([8]). *Let $T(G, \rho, H)$ be a generalized truncation of a r-regular graph G by a graph H of order r. The subgraphs of $T(G, \rho, H)$ induced by the sets $\{u_1^v, u_2^v, \ldots, u_r^v\}$, $v \in V(G)$, are non-overlapping isomorphic copies of H that partition the vertex set of the truncation, and the edges that connect different copies of H form a 1-factor of the truncation.*

Theorem 3.5 ([2]). *Let G be a finite (r, g)-graph with a vertex-neighborhood labeling ρ, and let H be a (r_0, g_0)-graph of order r. The generalized truncation graph $T(G, \rho, H)$ is a $(r_0 + 1)$-regular graph of girth not smaller than $min\{2g, g_0\}$, and if $g_0 \leq 2g$, then g_0 is the exact girth of $T(G, \rho, H)$.*

In what follows, we take H to be an (r, g, k)-graph and G to be a $(|V(H)|, g')$-graph. Since H is an induced subgraph of $T(G, \rho, H)$, it follows that $\chi(T(G, \rho, H)) \geq \chi(H) = k$. The last observation together with the previous known results about $T(G, \rho, H)$ give us the next remark.

Remark 3.6. Let H be a (r, g, k)-graph, and let G be a $(|V(H)|, g')$-graph, $g' \geq g/2$. Then

1. The graph $T(G, \rho, H)$ is $(r + 1)$-regular.
2. The girth of $T(G, \rho, H)$ is equal to g.
3. The chromatic number of $T(G, \rho, H)$ is at least k.

Theorem 3.7. *Let $k \geq 3$, let H be an (r, g, k)-graph of order h, and let G be a connected (h, g')-graph with $g' \geq g/2$. Then there is a function ρ such that $T(G, \rho, H)$ is a $(r + 1, g, k)$-graph.*

Proof. Let H be an (r, g, k)-graph of order h and let φ be a k-coloring of H. We may assume that for each chromatic class C_i, $|C_i| < h/2$, if not we take two copies of H, H_1 and H_2, color H_1 with φ and H_2 with a permutation of the colors of φ in order to obtain a graph and a coloring with the desired property.

Let $\mathcal{GH}$ be the set of all $T(G, \rho, H)$ graphs. Of all the graphs in $\mathcal{GH}$ and all possible k-colorings of the graph H with the property that $|C_i| < h/2$ for each $1 \leq i \leq k$, choose the pair such that the graph $G * H \in \mathcal{GH}$ has the minimum number of monochromatic edges (an edge is monochromatic if its vertices have the same color). Observe that since each copy of H is colored by φ, the coloring of $G * H$ may not be a proper coloring. Assume, for a contradiction, that $G * H$ has a monochromatic edge. Since each subgraph H_v is proper colored, this edge must be an external edge $\{u_i^\alpha, u_j^\beta\}$, w.l.o.g. $\varphi(u_i^\alpha) = \varphi(u_j^\beta) = 1$.

Claim 3.8. *Every external edge of H_α and H_β has at least one vertex of color 1.*

In order to state the Claim, We prove that if $\varphi(u_k^\alpha) \neq 1$ (or $\varphi(u_k^\beta) \neq 1$), then the external edge $\{u_k^\alpha, u_\ell^\gamma\}$ ($\{u_k^\beta, u_\ell^\gamma\}$ resp.) is such that $\varphi(u_\ell^\gamma) = 1$.

For a contradiction, suppose that $\varphi(u_\ell^\gamma) \neq 1$. Thus, we may swap the external edges of $\{u_k^\alpha\}$ and $\{u_i^\alpha\}$ and construct the graph

$$D = G * H \cup \Big\{\{u_i^\alpha, u_\ell^\gamma\}, \{u_k^\alpha, u_j^\beta\}\Big\} - \Big\{\{u_i^\alpha, u_j^\beta\}, \{u_k^\alpha, u_\ell^\gamma\}\Big\},$$

which is a graph in $\mathcal{GH}$ with lesser monochromatic edges than $G * H$, a contradiction, and the Claim follows.

Let $\varphi(u_x^\alpha) \neq 1$, and let $(u_y^\gamma) = N(u_x^\alpha) \setminus V(H_\alpha)$, by Claim 3.8, $\varphi(u_y^\gamma) = 1$. Let u_r^γ be a vertex such that $\varphi(u_r^\gamma) \neq 1$ and let $u_s^\delta = N(u_r^\gamma) \setminus V(H_\gamma)$. If $\varphi(u_s^\delta) \neq 1$, then consider the graph

$$D = G * H \cup \Big\{\{u_i^\alpha, u_\ell^\gamma\}, \{u_k^\alpha, u_j^\beta\}\Big\} - \Big\{\{u_i^\alpha, u_j^\beta\}, \{u_k^\alpha, u_\ell^\gamma)\}\Big\}.$$

Observe that $D \in \mathcal{GH}$ has the same number of monochromatic edges as $G * H$, and H_α has an external-edge with no vertex of color 1, contradicting Claim 3.8. Thus, if γ is a neighbor of α or a neighbor of β, then the external edges of H_γ must have a vertex of color 1. Since G is connected, we may apply this argument in each copy of H and so, every external edge must have a vertex of color 1, which is impossible, because, by hypothesis, there are more vertices which has color different from 1 than vertices of color 1.

Thus, $G * H$ has no monochromatic edges, φ is a proper coloring and, by Remark 3.6, $G * H$ is an $(r + 1, g, k)$-graph. □

As mentioned before, the Brinkmann graph B has order 21 and is an example of a $(4, 5, 4)$-cage. Using this result and Theorem 3.7, there is a generalized truncation $T(K_{22}, \rho, B)$ which is a $(5, 5, 4)$-graph. The order of this graph is $22(21) = 462$ which is a much better lower bound than the one obtained by item 1 of Corollary 2.4.

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