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NONLINEAR PHYSICS AND MECHANICS

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Application of the Kudryashov Method for Finding Exact Solutions of the Schamel – Kawahara Equation

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Recently, motivated by the interest in the problems of nonlinear dynamics of cylindrical shells, A. I. Zemlyanukhin *et al.* (Nonlinear Dyn, **98**, 185–194, 2019) established the so-called Schamel – Kawahara equation (SKE). The SKE generalizes the well-known nonlinear Schamel equation that arises in plasma physics problems, by adding the high-order dispersive terms from the Kawahara equation. This article presents families of new solutions to the Schamel – Kawahara model using the Kudryashov method. By performing the symbolic computation, we show that this method is a valuable and efficient mathematical tool for solving application problems modeled by nonlinear partial differential equations (NPDE).

Keywords: Schamel – Kawahara equation, Kudryashov method, exact solutions, nonlinear PDE

1. Introduction

Nonlinear partial differential equations (NPDE) provide mathematical models used in many areas of science and engineering to explain complex phenomena of many and very diverse prob-

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lems, for example, in electromagnetic wave theory, plasma physics, fluid mechanics, field theory, nonlinear optics, chemical kinetics, structural dynamics, stellar evolution, population dynamics, the evolution of financial flows, control theory, etc. Finding exact solutions to NPDEs has become a challenge among mathematics researchers and its applications because an exact solution provides essential information that will help describe the behavior of the phenomena modeled by the NPDE. As far as we know, there is no classical method that works to find exact solutions to any kind of NPDE. Therefore, the scientific community highly appreciates the investigation of new mathematical methodologies that provide exact solutions. Recently in [1], the author developed a new method to solve NPDE, named the Kudryashov method (KM), and it has been very useful to solve a great diversity of NPDEs of both integer and rational order that arise from applications of mathematics and many other disciplines [2–14].

Several models have been proposed to study nonlinear oscillations and waves for the structural stability of shells, rods, and plates [15, 16]. Especially, the propagation in bars and cylindrical shells is of interest since they are fitting for experimental research and have many and very diverse technological and engineering applications such as those in the modeling of airplanes, rockets, the design of gas and oil pipelines, and other similar constructions besides nanomaterial modeling [15–17]. Quite complicated and high-order equations or systems of equations are used to describe such waves. For instance, nonlinear stationary solitary waves that propagate with constant velocity without modifying their shape, the so-called solitons, are described by the Korteweg – de Vries (KdV) equation. The KdV equation is a fourth-order equation in the longitudinal displacement or a third-order equation in its derivative, which can be understood as a measure of the perturbation of the system [18, 19].

Although several extensions of the KdV equation exist, Schamel proposed an equation to study the propagation of solitary ion-acoustic waves with finite but small amplitude in a plasma [20]. This equation presents a stronger nonlinearity, having a term of the square root of the perturbation corresponding to a smaller width and a higher wave speed. On the other hand, the Kawahara equation was proposed in the study of magnetoacoustic waves in a cold plasma [21]. It also appropriately describes several phenomena observed in the wave propagation dynamics in water when the surface tension is not very large [22]. This equation contains a fifth-order dispersive term in addition to the third-order one in the KdV equation, which is added to balance the nonlinear effects in the studied cases. Among the solutions, traveling waves [23, 24] and solitary waves [25, 26] have been obtained for both equations.

Recently in [27], the propagation of longitudinal deformation waves in an inhomogeneous cylindrical shell was studied, in the case when the dependence of tension strength on deformation has a couple of nonlinear contributions, and in the regime when nonlinearity and dispersion are of the same order. For this case, a new NPDE named Schamel – Kawahara equation was obtained for such a system. It contains both the dispersive term of the fifth-order and the nonlinear term that contains the square root of the perturbation, in addition to the standard KdV terms. As far as we know, this new quasi-hyperbolic NPDE has only been studied in [27]. Therefore, it is imperative to implement some suitable method to obtain new solutions. The main objective of the present study is to obtain new solutions to the equation mentioned above by using the well-known Kudryashov method without the need for discretization, linearization, or truncation of the original model.

2. Governing equation

We will consider the Schamel – Kawahara equation which is a NPDE and whose dimensionless form is given by

$$u_t + c_1\sqrt{u}u_x + c_2u_{xxx} - c_3u_{xxxxx} + c_4uu_x = 0, \tag{2.1}$$

where $u(x,\, t)$ is the perturbation wave function that depends on the spatial variable x and on time t. The term u_t describes the temporal evolution of the wave propagation, c_i (with $i = 1,\, 2,\, 3,\, 4$) are nonzero constants and their meaning is as follows: c_1 is the activation trapping coefficient, c_2 and c_3 are dispersion coefficients, while c_4 is a convection coefficient.

The c_1 term corresponds to a stronger nonlinear factor than the c_4 term, the standard convective nonlinearity in the KdV equation. The Kawahara term of the fifth-order is that of the c_3 coefficient and is introduced to balance the nonlinearities induced by c_4. It should be noted that if in Eq. (2.1) $c_1 = 0$ the equation reduces to the Kawahara equation [21], if $c_3 = 0$ the equation is the Schamel – Korteweg – de Vries equation [19], while if $c_3 = 0$ and $c_4 = 0$ the equation is reduced to the well-known Schamel equation [20]. Note that all c_i are functions of physical and geometrical quantities that depend on the particular system to be modeled [24].

3. Brief description of the Kudryashov method

The Kudryashov method originally established in [1] provides a very useful algorithm to find exact solutions of NPDEs. Below we will briefly explain the steps of the method.

Consider the general nonlinear PDE given by

$$G(u,\, u_t,\, u_x,\, u_{xx},\, u_{xxx},\, \ldots) = 0. \tag{3.1}$$

Using the traveling wave variable change $u(x,\, t) = u(\xi)$ con $\xi = x - \omega t$, Eq. (3.1) becomes the ODE:

$$F(u,\, -\omega u_\xi,\, u_\xi,\, u_{\xi\xi},\, u_{\xi\xi\xi},\, \ldots) = 0. \tag{3.2}$$

To implement the Kudryashov method, it is useful to follow the following steps:

Step 1: We will assume that the exact solution of Eq. (3.2) has the form

$$u(\xi) = \sum_{i=0}^{N} a_iQ^i(\xi) = a_0 + a_1Q(\xi) + \cdots + a_NQ^N(\xi), \tag{3.3}$$

where each a_i is a constant with $a_N \neq 0$. The coefficients a_i will be algebraically determined, while the function Q is a solution of the Ricatti differential equation

$$\frac{dQ}{d\xi} = Q^2 - Q, \tag{3.4}$$

therefore, Q is given by

$$Q(\xi) = \frac{1}{1+e^{\xi}}. \tag{3.5}$$

Step 2: To find N, the upper bound of summation in (3.3), we substitute $u(z) = z^{-p}$ with $p > 0$ in equation (3.2) and, by comparing the two (or more) terms with the smallest powers, we find the value of N. That is, the maximum value of p is the pole order of Eq. (3.2) and we denote it as N.

Step 3: We consider $u(\xi)$ given in (3.3) and the necessary derivatives u_ξ, $u_{\xi\xi}$, $u_{\xi\xi\xi}$, … to substitute them in (3.2) and thus we will obtain the polynomial equation:

$$P[Q(\xi)] = 0. \tag{3.6}$$

Step 4: We select all the terms that have the same algebraic power of Q from the polynomial equation (3.6), set them equal to zero and obtain a system of algebraic equations with the set of unknowns $\{a_0, \ldots, a_N, \omega\}$. We can use some calculation software, such as Mathematica, to solve the system with the natural restrictions of the model and also considering that it is required that $a_N \neq 0$.

Step 5: With the results obtained in the previous step and considering Eq. (3.5) with Eq. (3.3), we obtain the possible exact solutions of Eq. (3.2) and therefore those of Eq. (3.1).

4. The Kudryashov method for SKE equation

Let us consider Eq. (2.1). Changing the variables $\xi = x - \omega t$, we obtain $u(x, t) = u(\xi)$, which:

$$u_t = -\omega u_\xi, \quad u_x = u_\xi, \quad \ldots, \quad u_{xxxxx} = u_{\xi\xi\xi\xi\xi}.$$

Substituting in Eq. (2.1), we obtain the following nonlinear ordinary differential equation:

$$-\omega u_\xi + c_1\sqrt{u}u_\xi + c_2 u_{\xi\xi\xi} - c_3 u_{\xi\xi\xi\xi\xi} + c_4 u u_\xi = 0. \tag{4.1}$$

Integrating once (4.1) with respect to ξ and canceling the constants of integration, we obtain

$$-\omega u + \frac{2}{3}c_1 u^{3/2} + c_2 u_{\xi\xi} - c_3 u_{\xi\xi\xi\xi} + \frac{c_4}{2}u^2 = 0. \tag{4.2}$$

Now let us consider $\omega = 2c_3\Omega$ with $c_3 \neq 0$ with which Eq. (4.2) is transformed to the equation

$$-\Omega u + \frac{c_1}{2c_3}u^{3/2} + \frac{c_4}{4c_3}u^2 + \frac{c_2}{2c_3}u_{\xi\xi} - \frac{1}{2}u_{\xi\xi\xi\xi} = 0, \tag{4.3}$$

which results in

$$-\Omega u + \frac{1}{3}\alpha u^{3/2} + \frac{1}{2}\beta u_{\xi\xi} - \frac{1}{2}u_{\xi\xi\xi\xi} + \frac{1}{4}\gamma u^2 = 0, \tag{4.4}$$

where

$$\alpha = \frac{c_1}{c_3}, \quad \beta = \frac{c_2}{c_3} \quad \text{and} \quad \gamma = \frac{c_4}{c_3}.$$

Finally, making the new variable change $\phi = \sqrt{u}$ with $\phi \geqslant 0$, Eq. (4.4) turns out to be the ordinary nonlinear differential equation:

$$-\Omega\phi^2 + \frac{1}{3}\alpha\phi^3 + \frac{1}{4}\gamma\phi^4 + \beta(\phi_\xi)^2 + \beta\phi\phi_{\xi\xi} - 3(\phi_{\xi\xi})^2 - 4\phi_\xi\phi_{\xi\xi\xi} - \phi\phi_{\xi\xi\xi\xi} = 0. \tag{4.5}$$

Considering that the nonlinear term of the highest algebraic power is ϕ^4 and the highest derivative term is $\phi\phi_{\xi\xi\xi\xi}$, we obtain the relation

$$4N = N + (N + 4),$$

from where $N = 2$. Therefore,

$$\phi(\xi) = a_0 + a_1 Q + a_2 Q^2, \tag{4.6}$$

and then the solution of Eq. (2.1) will be given by

$$u(x,\, t) = u(\xi) = \phi^2(\xi). \tag{4.7}$$

Knowing *a priori* that ϕ will be given as the polynomial in the indeterminate Q of Eq. (4.6), we calculate its derivatives up to fourth order using the formulas previously obtained in [1] for the case $N = 2$, and obtain

$$\phi_\xi = 2a_2Q^3 + (a_1 - 2a_2)Q^2 - a_1Q, \tag{4.8}$$

$$\phi_{\xi\xi} = 6a_2Q^4 + (a_1 - 10a_2)Q^3 + (4a_2 - 3a_1)Q^2 + a_1Q, \tag{4.9}$$

$$\phi_{\xi\xi\xi} = 24a_2Q^5 + (6a_1 - 54a_2)Q^4 + (38a_2 - 12a_1)Q^3 + (7a_1 - 8a_2)Q^2 - a_1Q, \tag{4.10}$$

$$\begin{aligned}\phi_{\xi\xi\xi\xi} = {} & 120a_2Q^6 + (24a_1 - 336a_2)Q^5 - (60a_1 - 330a_2)Q^4 + (50a_1 - 130a_2)Q^3 + \\ & + (16a_2 - 15a_1)Q^2 + a_1Q.\end{aligned} \tag{4.11}$$

Substituting Eq. (4.6) and Eqs. (4.8)–(4.11) into Eq. (4.5), we get:

$$\begin{aligned}
& Q^8\left(\frac{a_2^4\gamma}{4} - 420a_2^2\right) + Q^7\left(a_1a_2^3\gamma + 1320a_2^2 - 360a_1a_2\right) + \\
& + Q^6\left(\frac{\alpha a_2^3}{3} + 10a_2^2\beta + a_0a_2^3\gamma + \frac{3}{2}a_1^2a_2^2\gamma - 1510a_2^2 - 120a_0a_2 + 1080a_1a_2 - 60a_1^2\right) + \\
& + Q^5(\alpha a_2^2a_1 + 12a_2a_1\beta - 18a_2^2\beta + a_2a_1^3\gamma + 3a_0a_2^2a_1\gamma + 168a_1^2 - 24a_0a_1 - \\
& - 1164a_2a_1 + 738a_2^2 + 336a_0a_2) + Q^4\Bigg(\alpha a_2a_1^2 + \alpha a_0a_2^2 + 3a_1^2\beta - 21a_2a_1\beta + 8a_2^2\beta + 6a_0a_2\beta + \\
& + \frac{a_1^4\gamma}{4} + 3a_0a_2a_1^2\gamma + \frac{3}{2}a_0^2a_2^2\gamma - a_2^2\Omega - 165a_1^2 + 60a_0a_1 + 525a_2a_1 - 128a_2^2 - 330a_0a_2\Bigg) + \\
& + Q^3\Bigg(\frac{\alpha a_1^3}{3} + 2\alpha a_0a_2a_1 - 5a_1^2\beta + 2a_0a_1\beta + 9a_2a_1\beta - 10a_0a_2\beta + a_0a_1^3\gamma + 3a_0^2a_2a_1\gamma - \\
& - 2a_2a_1\Omega + 65a_1^2 - 50a_0a_1 - 81a_2a_1 + 130a_0a_2\Bigg) + Q^2\Bigg(\alpha a_2a_0^2 + \alpha a_1^2a_0 - 3a_1a_0\beta + \\
& + 4a_2a_0\beta + 2a_1^2\beta + a_2a_0^3\gamma + \frac{3}{2}a_1^2a_0^2\gamma - 2a_2a_0\Omega - a_1^2\Omega + 15a_1a_0 - 16a_2a_0 - 8a_1^2\Bigg) + \\
& + Q\left(\alpha a_1a_0^2 + a_1a_0\beta + a_1a_0^3\gamma - 2a_1a_0\Omega - a_0a_1\right) + \frac{\alpha a_0^3}{3} + \frac{a_0^4\gamma}{4} - a_0^2\Omega = 0.
\end{aligned} \tag{4.12}$$

Collecting the coefficients of the same power and setting them equal to zero, we obtain the system of algebraic equations with the set of unknowns $\{a_0,\, a_1,\, a_2,\, \Omega\}$:

$$\begin{aligned}
Q^0: &\quad \frac{\alpha a_0^3}{3} + \frac{a_0^4\gamma}{4} - a_0^2\Omega = 0, \\
Q^1: &\quad \alpha a_1a_0^2 + a_1a_0\beta + a_1a_0^3\gamma - 2a_1a_0\Omega - a_0a_1 = 0, \\
Q^2: &\quad \alpha a_2a_0^2 + \alpha a_1^2a_0 - 3a_1a_0\beta + 4a_2a_0\beta + 2a_1^2\beta + a_2a_0^3\gamma + \frac{3}{2}a_1^2a_0^2\gamma - \\
&\quad -2a_2a_0\Omega - a_1^2\Omega + 15a_1a_0 - 16a_2a_0 - 8a_1^2 = 0,
\end{aligned}$$

$$Q^3: \quad \frac{\alpha a_1^3}{3} + 2\alpha a_0 a_2 a_1 - 5a_1^2\beta + 2a_0 a_1\beta + 9a_2 a_1\beta - 10a_0 a_2\beta + a_0 a_1^3\gamma + 3a_0^2 a_2 a_1\gamma - \\ -2a_2 a_1\Omega + 65a_1^2 - 50a_0 a_1 - 81a_2 a_1 + 130a_0 a_2 = 0,$$

$$Q^4: \quad \alpha a_2 a_1^2 + \alpha a_0 a_2^2 + 3a_1^2\beta - 21a_2 a_1\beta + 8a_2^2\beta + 6a_0 a_2\beta + \frac{a_1^4\gamma}{4} + 3a_0 a_2 a_1^2\gamma + \frac{3}{2}a_0^2 a_2^2\gamma - \\ -a_2^2\Omega - 165a_1^2 + 60a_0 a_1 + 525a_2 a_1 - 128a_2^2 - 330a_0 a_2 = 0,$$

$$Q^5: \quad \alpha a_2^2 a_1 + 12a_2 a_1\beta - 18a_2^2\beta + a_2 a_1^3\gamma + 3a_0 a_2^2 a_1\gamma + 168a_1^2 - 24a_0 a_1 - 1164a_2 a_1 + \\ +738a_2^2 + 336a_0 a_2 = 0,$$

$$Q^6: \quad \frac{\alpha a_2^3}{3} + 10a_2^2\beta + a_0 a_2^3\gamma + \frac{3}{2}a_1^2 a_2^2\gamma - 1510a_2^2 - 120a_0 a_2 + 1080a_1 a_2 - 60a_1^2 = 0,$$

$$Q^7: \quad a_1 a_2^3\gamma + 1320a_2^2 - 360a_1 a_2 = 0,$$

$$Q^8: \quad \frac{a_2^4\gamma}{4} - 420a_2^2 = 0.$$

Solving the system of algebraic equations above with the restriction of the method $a_2 \neq 0$, we obtain, with the help of *Mathematica* software, the following families of results:

Family 1: With $\alpha = \frac{1}{2}\sqrt{\frac{15\gamma}{7}}(\beta - 13)$, $\beta \in \mathbb{R}$, $\beta \neq 13$ and $\gamma > 0$:

$$a_0 = \frac{\sqrt{105\gamma}(\beta - 13) - 14\alpha}{39\gamma}, \quad a_1 = 4\sqrt{\frac{105}{\gamma}}, \quad a_2 = -4\sqrt{\frac{105}{\gamma}}, \quad \Omega = 2(\beta - 4).$$

Using the Ansatz given by Eq. (4.6), we obtain the following traveling-wave solution of Eq. (4.5):

$$\phi_1(\xi) = \frac{\sqrt{105\gamma}(\beta - 13) - 14\alpha}{39\gamma} + 4\sqrt{\frac{105}{\gamma}}\left(\frac{1}{1 + e^{\xi}}\right) - 4\sqrt{\frac{105}{\gamma}}\left(\frac{1}{1 + e^{\xi}}\right)^2, \tag{4.13}$$

considering that $\alpha = \frac{c_1}{c_3}$, $\beta = \frac{c_2}{c_3}$, $\gamma = \frac{c_4}{c_3}$, $\omega = 2c_3\Omega$ and $u = \phi^2$, we obtain the following solution for Eq. (2.1):

$$u_1(x, t) = \left[A_1 + B_1\left(\frac{1}{1 + e^{(x - \omega_1 t)}}\right) - B_1\left(\frac{1}{1 + e^{(x - \omega_1 t)}}\right)^2\right]^2 = \\ = \left[\frac{A_1 + (2A_1 + B_1)e^{(x - \omega_1 t)} + A_1 e^{2(x - \omega_1 t)}}{(1 + e^{(x - \omega_1 t)})^2}\right]^2, \tag{4.14}$$

where

$$A_1 = \frac{\sqrt{\frac{105c_4}{c_3}}\left(\frac{c_2 - 13c_3}{c_3}\right) - \frac{14c_1}{c_3}}{\frac{39c_4}{c_3}}, \quad B_1 = 4\sqrt{\frac{105c_3}{c_4}} \quad \text{and} \quad \omega_1 = 4(c_2 - 4c_3) \quad \text{with} \quad c_3 c_4 > 0.$$

Family 2: With $\alpha = \frac{1}{2}\sqrt{\frac{15\gamma}{7}}(\beta - 13)$, $\beta \in \mathbb{R}$, $\beta \neq 13$ and $\gamma > 0$:

$$a_0 = 0, \quad a_1 = 4\sqrt{\frac{105}{\gamma}}, \quad a_2 = -4\sqrt{\frac{105}{\gamma}}, \quad \Omega = 2(\beta - 4).$$

Using the Ansatz given by Eq. (4.6), we obtain the following traveling-wave solution of Eq. (4.5):

$$\phi_2(\xi) = 4\sqrt{\frac{105}{\gamma}}\left(\frac{1}{1+e^{\xi}}\right) - 4\sqrt{\frac{105}{\gamma}}\left(\frac{1}{1+e^{\xi}}\right)^2, \tag{4.15}$$

from which finally the corresponding solution for Eq. (2.1) is

$$u_2(x,\,t) = \left[B_2\left(\frac{1}{1+e^{(x-\omega_2 t)}}\right) - B_2\left(\frac{1}{1+e^{(x-\omega_2 t)}}\right)^2\right]^2 = \left[\frac{B_2 e^{(x-\omega_2 t)}}{(1+e^{(x-\omega_2 t)})^2}\right]^2, \tag{4.16}$$

where

$$B_2 = 4\sqrt{\frac{105c_3}{c_4}} \quad \text{and} \quad \omega_2 = 4(c_2 - 4c_3) \quad \text{with} \quad c_3c_4 > 0.$$

Family 3: With $\alpha = -\frac{1}{2}\sqrt{\frac{15\gamma}{7}}(\beta - 13)$, $\beta \in \mathbb{R}$, $\beta \neq 13$ and $\gamma > 0$:

$$a_0 = 0, \quad a_1 = -4\sqrt{\frac{105}{\gamma}}, \quad a_2 = 4\sqrt{\frac{105}{\gamma}}, \quad \Omega = 2(\beta - 4).$$

Using the Ansatz given by Eq. (4.6), we obtain the following traveling-wave solution of Eq. (4.5):

$$\phi_3(\xi) = -4\sqrt{\frac{105}{\gamma}}\left(\frac{1}{1+e^{\xi}}\right) + 4\sqrt{\frac{105}{\gamma}}\left(\frac{1}{1+e^{\xi}}\right)^2. \tag{4.17}$$

from the above, the corresponding solution for Eq. (2.1) is

$$u_3(x,\,t) = \left[-B_3\left(\frac{1}{1+e^{(x-\omega_3 t)}}\right) + B_3\left(\frac{1}{1+e^{(x-\omega_3 t)}}\right)^2\right]^2 = \left[\frac{-B_3 e^{(x-\omega_3 t)}}{(1+e^{(x-\omega_3 t)})^2}\right]^2, \tag{4.18}$$

where

$$B_3 = 4\sqrt{\frac{105c_3}{c_4}} \quad \text{and} \quad \omega_3 = 4(c_2 - 4c_3) \quad \text{with} \quad c_3c_4 > 0.$$

Family 4: With $\alpha = \alpha$, $\beta = \frac{1}{15}\left[195 + 2\alpha\sqrt{\frac{105}{\gamma}}\right]$ and $\gamma > 0$:

$$a_0 = 0, \quad a_1 = 4\sqrt{\frac{105}{\gamma}}, \quad a_2 = -4\sqrt{\frac{105}{\gamma}}, \quad \Omega = 7\beta - 73 - \frac{2\alpha}{3}\sqrt{\frac{105}{\gamma}}.$$

Using the Ansatz given by Eq. (4.6), we obtain the following traveling-wave solution of Eq. (4.5):

$$\phi_4(\xi) = 4\sqrt{\frac{105}{\gamma}}\left(\frac{1}{1+e^{\xi}}\right) - 4\sqrt{\frac{105}{\gamma}}\left(\frac{1}{1+e^{\xi}}\right)^2, \tag{4.19}$$

from the above, the corresponding solution for Eq. (2.1) is

$$u_4(x,\,t) = \left[B_4\left(\frac{1}{1+e^{(x-\omega_4 t)}}\right) - B_4\left(\frac{1}{1+e^{(x-\omega_4 t)}}\right)^2\right]^2 = \left[\frac{B_4 e^{(x-\omega_4 t)}}{(1+e^{(x-\omega_4 t)})^2}\right]^2, \tag{4.20}$$

where

$$B_4 = 4\sqrt{\frac{105c_3}{c_4}} \quad \text{and} \quad \omega_4 = 36c_3 - \frac{32}{15}c_1\sqrt{\frac{105c_3}{c_4}} \quad \text{with} \quad c_3c_4 > 0.$$

Family 5: With $\alpha = \alpha$, $\beta = \frac{1}{15}\left[195 - 2\alpha\sqrt{\frac{105}{\gamma}}\right]$ and $\gamma > 0$:

$$a_0 = 0, \quad a_1 = -4\sqrt{\frac{105}{\gamma}}, \quad a_2 = 4\sqrt{\frac{105}{\gamma}}, \quad \Omega = 7\beta - 73 + \frac{2\alpha}{3}\sqrt{\frac{105}{\gamma}}.$$

Using the Ansatz given by Eq. (4.6), we obtain the following traveling-wave solution of Eq. (4.5):

$$\phi_5(\xi) = -4\sqrt{\frac{105}{\gamma}}\left(\frac{1}{1+e^{\xi}}\right) + 4\sqrt{\frac{105}{\gamma}}\left(\frac{1}{1+e^{\xi}}\right)^2, \tag{4.21}$$

from the above, the corresponding solution for Eq. (2.1) is

$$u_5(x,\, t) = \left[-B_5\left(\frac{1}{1+e^{(x-\omega_5 t)}}\right) + B_5\left(\frac{1}{1+e^{(x-\omega_5 t)}}\right)^2\right]^2 = \left[\frac{-B_5e^{(x-\omega_5 t)}}{(1+e^{(x-\omega_5 t)})^2}\right]^2, \tag{4.22}$$

where

$$B_5 = 4\sqrt{\frac{105c_3}{c_4}} \quad \text{and} \quad \omega_5 = 36c_3 + \frac{32}{15}c_1\sqrt{\frac{105c_3}{c_4}} \quad \text{with} \quad c_3c_4 > 0.$$

Family 6: With $\gamma \neq 0$ and $\beta = 13$:

$$a_0 = 0, \quad a_1 = -4\sqrt{\frac{105}{\gamma}}, \quad a_2 = 4\sqrt{\frac{105}{\gamma}}, \quad \Omega = 18.$$

Using the Ansatz given by Eq. (4.6), we obtain the following traveling-wave solution of Eq. (4.5):

$$\phi_6(\xi) = -4\sqrt{\frac{105}{\gamma}}\left(\frac{1}{1+e^{\xi}}\right) + 4\sqrt{\frac{105}{\gamma}}\left(\frac{1}{1+e^{\xi}}\right)^2, \tag{4.23}$$

from which finally the corresponding solution for Eq. (2.1) is

$$u_6(x,\, t) = \left[-B_6\left(\frac{1}{1+e^{(x-\omega_6 t)}}\right) + B_6\left(\frac{1}{1+e^{(x-\omega_6 t)}}\right)^2\right]^2 = \left[\frac{-B_6e^{(x-\omega_6 t)}}{(1+e^{(x-\omega_6 t)})^2}\right]^2, \tag{4.24}$$

where

$$B_6 = 4\sqrt{\frac{105c_3}{c_4}} \quad \text{and} \quad \omega_6 = 36c_3 \quad \text{with} \quad c_3c_4 > 0.$$

5. Graphical presentation of solutions

In this section we will show graphically some solutions of Eq. (2.1) for different values of the coefficients c_1, c_2, c_3 and c_4 as well as the corresponding 2D plots for some values of t illustrating how the traveling wave evolves.

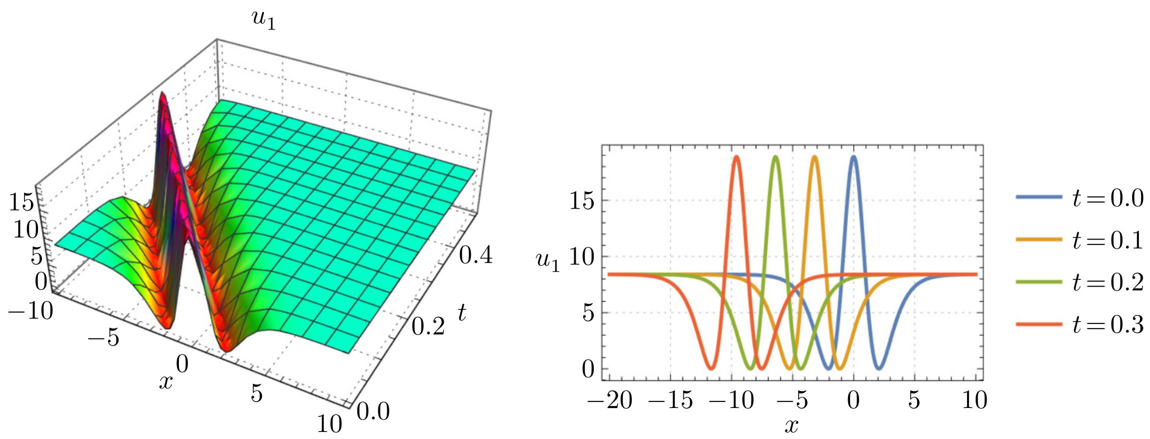

Fig. 1. Using Eq. (4.14): Solitary wave solutions for Eq. (2.1) with $c_1 = 2.0$, $c_2 = -2.0$, $c_3 = 1.5$ and $c_4 = 3.0$ (left). 2D plot of the exact solution u_1 for $t = 0.0$, $t = 0.1$, $t = 0.2$ and $t = 0.3$ (right)

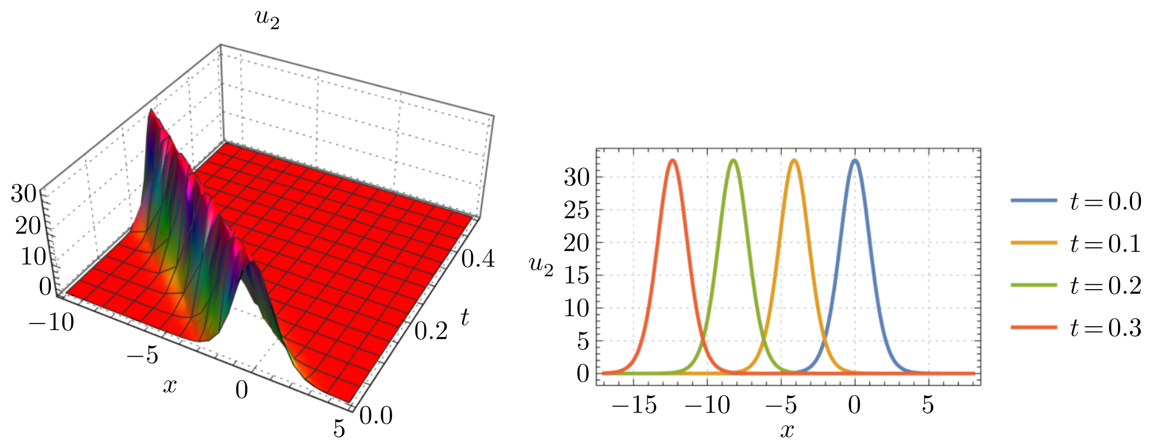

Fig. 2. Using Eq. (4.16): Solitary wave solutions for Eq. (2.1) with $c_1 = 1.3$, $c_2 = 2.1$, $c_3 = -3.1$ and $c_4 = -10.0$ (left). 2D plot of the exact solution u_2 for $t = 0.0$, $t = 0.1$, $t = 0.2$ and $t = 0.3$ (right)

(a) For family 1: Let us consider Eq. (2.1) for the values of the coefficients $c_1 = 2.0$, $c_2 = -2.0$, $c_3 = 1.5$ and $c_4 = 3.0$. Figure 1 shows the 3D and 2D W-shaped soliton solutions of Eq. (4.14).

(b) For family 2: Let us consider Eq. (2.1) for the values of the coefficients $c_1 = 1.3$, $c_2 = 2.1$, $c_3 = -3.1$ and $c_4 = -10.0$. Figure 2 shows the 3D and 2D bright soliton solutions of Eq. (4.16).

(c) For family 3: Let us consider Eq. (2.1) for the values of the coefficients $c_1 = 2.5$, $c_2 = 3.5$, $c_3 = 0.2$ and $c_4 = 0.5$. Figure 3 shows the 3D and 2D bright soliton solutions of Eq. (4.18).

(d) For family 4: Let us consider Eq. (2.1) for the values of the coefficients $c_1 = 4.8$, $c_2 = 1.8$, $c_3 = 3.1$ and $c_4 = 5.5$. Figure 4 shows the 3D and 2D bright soliton solutions of Eq. (4.20).

(e) For family 5: Let us consider Eq. (2.1) for the values of the coefficients $c_1 = 0.01$, $c_2 = 1.4$, $c_3 = -0.2$ and $c_4 = -6.2$. Figure 5 shows the 3D and 2D bright soliton solutions of Eq. (4.22).

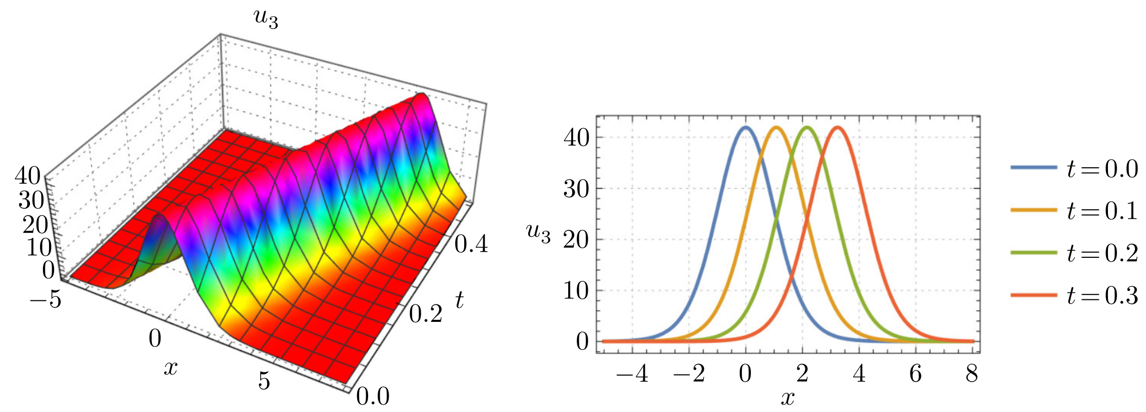

Fig. 3. Using Eq. (4.18): Solitary wave solutions for Eq. (2.1) with $c_1 = 2.5$, $c_2 = 3.5$, $c_3 = 0.2$ and $c_4 = 0.5$ (left). 2D plot of the exact solution u_3 for $t = 0.0$, $t = 0.1$, $t = 0.2$ and $t = 0.3$ (right)

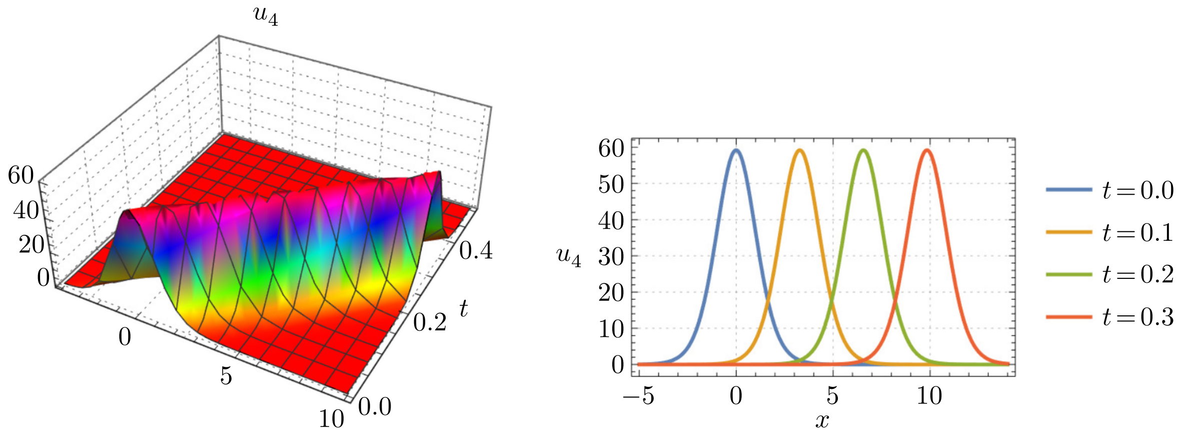

Fig. 4. Using Eq. (4.20): Solitary wave solutions for Eq. (2.1) with $c_1 = 4.8$, $c_2 = 1.8$, $c_3 = 3.1$ and $c_4 = 5.5$ (left). 2D plot of the exact solution u_4 for $t = 0.0$, $t = 0.1$, $t = 0.2$ and $t = 0.3$ (right)

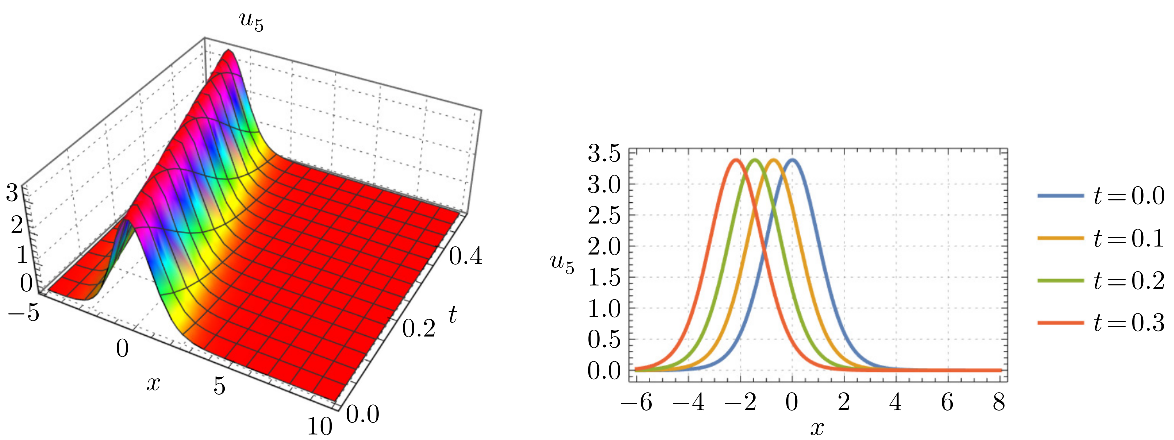

Fig. 5. Using Eq. (4.22): Solitary wave solutions for Eq. (2.1) with $c_1 = 0.01$, $c_2 = 1.4$, $c_3 = -0.2$ and $c_4 = -6.2$ (left). 2D plot of the exact solution u_5 for $t = 0.0$, $t = 0.1$, $t = 0.2$ and $t = 0.3$ (right)

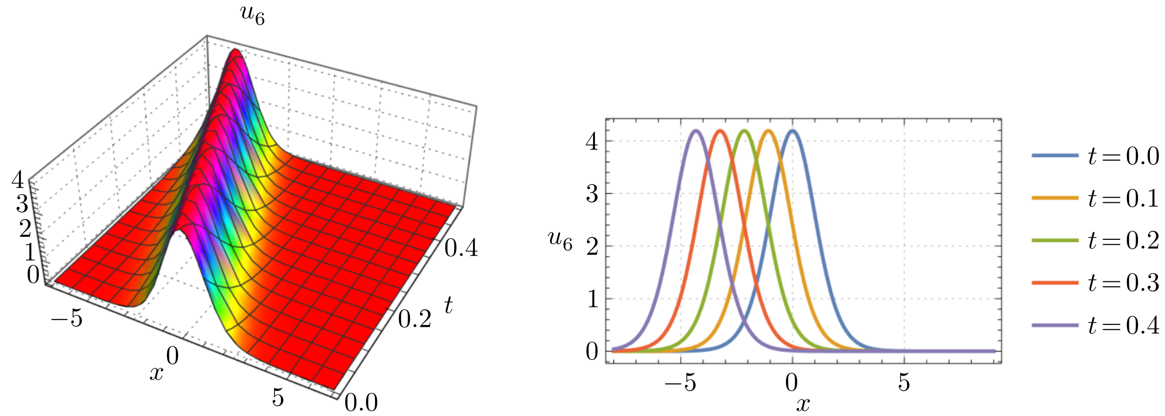

Fig. 6. Using Eq. (4.24): Solitary wave solutions for Eq. (2.1) with $c_1 = 3.9$, $c_2 = 4.4$, $c_3 = -0.3$ and $c_4 = -7.5$ (left). 2D plot of the exact solution u_6 for $t = 0.0$, $t = 0.1$, $t = 0.2$, $t = 0.3$ and $t = 0.4$ (right)

(f) For family 6: Let us consider Eq. (2.1) for the values of the coefficients $c_1 = 3.9$, $c_2 = 4.4$, $c_3 = -0.3$ and $c_4 = -7.5$. Figure 6 shows the 3D and 2D bright soliton solutions of Eq. (4.24).

From the above figures, one can see that the solutions obtained possess the W-shaped soliton solutions and the bright soliton solutions of Eq. (2.1). In addition, these figures show the behavior of longitudinal wave propagation in the cylindrical shell which give some perspective how the behavior solutions are produced.

Remark 1. The exact solutions (4.16), (4.18), (4.20), (4.22) and (4.24), which we obtained for the Schamel–Kawahara equation, have the same traveling wave structure as those found by the geometric series method using Padé approximations in [27].

6. Conclusions

Many and very diverse problems in engineering and science require the study of waves and oscillations to analyze structural stability. Recently, it was shown that the propagation of longitudinal waves in deformable media with axial symmetry follows the Schamel–Kawahara equation. The SKE is a generalization of the KdV equation that contains a nonlinear term associated with the Schamel equation and a dispersive high-order term related to the Kawahara equation, which was used to model perturbations in cylindrical shells [28]. Therefore, it is imperative to have solutions to this equation for this kind of propagation.

In this paper, a new set of families of traveling wave solutions of the Schamel–Kawahara equation has been obtained using the Kudryashov method. The Kudryashov method is a powerful tool for dealing with dispersive higher-order NPDEs and is easy to implement. It can be applied to a wide variety of NPDEs arising in different branches of science and engineering to explore complex nonlinear systems analytically. It provides a simple algorithm to find exact solitary wave solutions by reducing the problem to an algebraic one.

The six families were found to show similar behaviors. Five solutions gave bright soliton-like solutions, and one a W-type wave, for specific values of the parameters. These are shown in Figs. 1–6. In general, for each problem, the parameters will depend on the geometrical and physical characteristics of the system, such as thickness, length, rib spacing, Young's modulus, moments of inertia, etc. Hence, one could look for the values of a specific system and see which of these families provide real solutions to the problem.

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