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Nonrelativistic Boson stars as N-body quantum systems

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In this work, we show that the structural configuration of a collection of generic nonrelativistic bosons forming a gravitationally bound Bose–Einstein condensate can be interpreted as a nonrelativistic boson star. With the approach followed in this work, we can analyze with a concise and straightforward procedure the equilibrium properties of nonrelativistic boson stars viewed as a Bose–Einstein condensate. The system's behavior is obtained by analyzing its fundamental constituent properties, i.e. the associated single-particle properties. Additionally, we show that by expressing the corresponding Newtonian gravitational potential, under certain circumstances, as a harmonic oscillator potential one, we can describe the conditions in which the nonrelativistic boson star can form equilibrium configurations. In order to analyze the boson star's structural configuration, we employ four different *ansätzes* commonly used in the literature. These *ansätzes* allow us to compare the cloud's structural properties of the boson star, which leads us to obtain several gravitational equilibrium configurations from compact objects matching the size of typical stars to gigantic systems comparable to the size of galaxy cluster dark matter halos. Finally, we show that these *ansätzes* predict, qualitatively speaking, the same structural and gravitational equilibrium configurations for different values of the parameters involved.

Keywords: Boson stars; dark matter; Bose–Einstein condensates.

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1. Introduction

Bose–Einstein condensates (BECs) play a fascinating and essential role in modern physics, relating many models spreading from microscopic well-proved behavior of ultracold quantum gases to galactic and cosmological scales. Nevertheless, there is an issue not well understood in this scenario that deserves more in-depth study, i.e. the nontrivial conditions in which scalar fields can form BECs.[1–8] However, it seems to be that scalar fields in the form of a BEC formed by generic bosons can describe the basic properties of dark matter (DM) in the universe.[9–15] According to this line of thought, DM consists of a particular type of spin-zero bosons, such as ultralight scalar field dark matter or fuzzy dark matter, weakly interacting massive particles, axions, etc. (depending on the specific model under consideration) which have not yet been observed. The bosonic character of these particles, by using the theory of relativistic Bose gases,[16,17] also opens the door for the existence of scalar field dark matter in the form of BECs.[18,19]

Complementary to the ideas mentioned above, some particular theoretical objects can be formed in a very similar manner. In some circumstances, a system of *generic bosons* can form gravitationally bound BECs leading to macroscopic objects, the so-called boson stars (BSs).[20–23] On the one hand, some research lines suggest that BSs could be alternative candidates for black holes in the center of galaxies. Here it is important to mention that, related to the above ideas, some candidates knowing as a gravastar, first proposed by Mazur and Mottola,[24] could play also the role as an alternative object to black holes in the center of galaxies,[24,25] see also Ref. 26 for an interesting review on these ideas. Thus, the concept of BS as a gravastar (also seen as a gravitational BEC) suggests a solid and viable alternative scenario to those with a black hole at the center of galaxies. The gravitational condensate stars, or gravastars, are an extension of the concept of a Bose–Einstein condensate to gravitational systems, i.e. a cold, compact object of total arbitrary mass M, with an inner de Sitter in a condensate phase and an outer Schwarzschild geometry. It is essentially the same concept behind the relativistic BS's structure seen as a BEC, which is remarkable.

On the other hand, it is generally accepted that the fundamental constituents of BSs are some generic scalars formed as a BEC.[23,27] This last assertion opens up the opportunity to describe these objects with BEC's formalism, which enriches the analysis. There are numerous papers on boson stars in the literature; some treat the boson star as described by a single wave function, even comparing it to BECs. This wave function has been analyzed through various methods; for example, there are some papers by Eby *et al.*[28,29] Indeed, we must mention that the study of BSs and its interpretation as BECs has been extensively analyzed.[30] Although these objects have not been observed yet, their behavior and structural properties lead us to think that these systems are highly related to scalar dark matter clouds in the universe. There is a *zoo* of these objects (BSs) in the literature, basically characterized according to their dynamical behavior.[27] For instance, BSs could lie in the relativistic regime or not; they can also be characterized concerning the type of self-interactions within the system,[27] etc. The structural properties of the BS's are pretty interesting. For instance, Heisenberg's Uncertainty Principle provides pressure support in order to get a stable object. The size of the BS ranges from giant to very compact objects depending on the nature of the involved functional interactions among the system's constituents. The equations that govern the dynamics of the BS depend on the regime in which it is working. The most general case is the relativistic regime, where the Klein–Gordon equation coupled to gravity is used through the Ruffini–Bonazzola (RB) formalism. When the field is weakly coupled to gravity and the ground state is nonrelativistic, i.e. the binding energy is much lower than its mass, it is possible to approximate the Klein–Gordon equation as an Schrödinger-like equation with a self-interaction term that must be small enough to be consistent with the approximation. This is the so-called *Gross–Pitaevskii–Poisson equation* (GPP), see for instance Refs. 31 and 32 and references therein. For instance, in the case of an axion star (boson particles that are dark matter candidates), one can match the dynamics of the field in the nonrelativistic regime by expanding axion potential to be of the form $m^2|\phi|^2 + \lambda|\phi|^4$, being m the mass parameter and λ the term related to self-interactions. Physically, this case corresponds to a dilute axion star that is a stable solution of the GPP equation with the energy density well below the confinement QCD scale (1 GeV), so the nonrelativistic approximation adequately describes this system.[33,34] Clearly, more general interactions within the system can be described with the extensions of the Gross–Pitaevskii equation. However, within this approximation, it is possible to have a criterion of structural characterization of the BS, for instance, its size, stability, etc., by using (under certain conditions) the basic formalism behind usual laboratory BECs. Although, strictly speaking, the Gross–Pitaevskii equation is an approximated equation valid for systems at zero temperature, the predictions made by the Gross–Pitaevskii equation are a good approximation for temperatures $T < T_c$, where T_c is the condensation temperature of the system. Such an equation can be used to analyze diluted weakly interacting systems' properties and when the number of particles is large enough for the condensed phase. However, when the corresponding *Gross–Pitaevskii–Poisson equation* is used to study the structural properties of the BS, the N-particles that constitute the system are analyzed as a single particle(-field). In other words, the nature of the phase transition provides the advantage to reduce the analysis of the N-body system to analyze the dynamics of a single body(-field) as in usual BECs. For this reason, we call the field appearing in the *Gross–Pitaevskii–Poisson equation*, the order parameter. The order parameter contains the information of the N-particles forming the condensed phase, and due to the highly correlated behavior of the BEC, this system behaves as a single entity.

In this work, we show that a collection of weakly interacting generic bosons that form gravitationally bound BECs can describe the nonrelativistic behavior of a BS. In other words, we describe some structural properties related to the BS through the quantum properties of its fundamental constituents, i.e. the properties of a single particle. This paper is organized as follows. In Sec. 2, we study the fundamental properties of the BS viewed as a collection of bosons starting from the single-particle description. We assume that the system behaves as a BEC. Also, we describe the approximation in which the Newtonian gravitational potential can be expressed as a harmonic oscillator potential that we interpreted as the trapping potential, like in the usual laboratory BECs. In Sec. 3, we analyze the relevant structural functions that characterize the ground state of the system in order to obtain criteria of stability upon the BS. In Sec. 4, we analyze the equilibrium conditions upon several systems at different scales, and also describe some insights related to the corresponding equation of state (EoS). Finally, in Sec. 5, we present a discussion, conclusions, and outlook.

2. *N*-Body Quantum System as a Nonrelativistic Boson Star

As was mentioned in the introduction, the basic constituents of BSs are scalar particles (or spin zero-bosons), probably in the form of a BEC. In this section, we analyze the nonrelativistic BSs behavior as a collection of bosons interpreted as a quantum N-body system in order to analyze some relevant properties associated with the bosonic cloud viewed as a BEC. In this aim, we define the following N-body Hamiltonian, which describes our nonrelativistic BS

$$\hat{H} = -\frac{\hbar^2}{2m_\phi}\sum_{\delta,\gamma}\langle\delta|\nabla^2|\gamma\rangle\hat{a}^\dagger_\delta\hat{a}_\gamma + \frac{1}{2}\sum_{\delta,\gamma,\mu,\nu}\langle\delta,\gamma|V_{int}|\mu,\nu\rangle\hat{a}^\dagger_\delta\hat{a}_\gamma\hat{a}^\dagger_\mu\hat{a}_\nu + \sum_{\delta,\gamma}\langle\delta|V_g|\gamma\rangle\hat{a}^\dagger_\delta\hat{a}_\gamma, \tag{1}$$

where m_ϕ is the mass of the generic boson particle and $V_{\rm int}$ is the potential that describes the system's interactions. Moreover, we have also inserted in the Hamiltonian equation (1) the contributions of the gravitational potential V_g. Additionally, the operators $\hat{a}$ and $\hat{a}^\dagger$, correspond to the creation and annihilation operators for bosons, satisfying the usual canonical commutation relations

$$[\hat{a}_\mu, \hat{a}^\dagger_\nu] = \delta_{\mu\nu}, \quad [\hat{a}_\mu, \hat{a}_\nu] = [\hat{a}^\dagger_\mu, \hat{a}^\dagger_\nu] = 0. \tag{2}$$

As was mentioned above, the term, $V_{\rm int}$ denotes the interparticle potential, which will be assumed as $V_{\rm int} \equiv U_0 = \frac{4\pi\hbar^2}{m_\phi}a$, with a the s-wave scattering length, i.e.

at temperatures below the condensation temperature, only two-body interactions are taken into account. In other words, the system is diluted enough and fulfills the condition $\rho|a|^3 \ll 1$, where ρ is the density of particles.[35–38] Additionally, V_g depicts the contributions of the gravitational potential within the BS that we will analyze later in the paper.

Note that in the corresponding N-body Hamiltonian equation (1), we have the following terms:

$$\langle \delta \boldsymbol{\nabla}^2 | \gamma \rangle = \int d^3 r u_\delta^*(\mathbf{r}) \nabla^2 u_\gamma(\mathbf{r}), \tag{3}$$

$$\langle \delta, \gamma | V_{\text{int}} | \mu, \nu \rangle = \int \int d^3 r_1 d^3 r_2 u_\delta^*(\mathbf{r}_1) u_\gamma^*(\mathbf{r}_2) U_0 u_\mu(\mathbf{r}_2) u_\nu(\mathbf{r}_1), \tag{4}$$

$$\langle \delta | V_g | \gamma \rangle = \int d^3 r u_\delta^*(\mathbf{r}) V_g u_\gamma(\mathbf{r}), \tag{5}$$

where $\{u_\epsilon(\mathbf{r})\}$ is a set of single-particle functions. Strictly speaking, we must mention here that in order to calculate the corresponding total energy in Eq. (1), we have to solve the corresponding equation of motion for a single particle to obtain the single-particle set of functions $\{u_\epsilon(\mathbf{r})\}$ for the ground state. Formally, the corresponding wave function associated with the system's ground state would be the solution associated with the Gross–Pitaevskii equation coupled with gravity in our scenario, which describes the ground state properties of the system at temperatures $T < T_c$. However, as we will see later in the paper, for simplicity, we are able to employ an accurate expression for the total energy of the cloud (or the BS) that can be obtained by using, as usual, several *ansätzes* commonly used in the literature that describes the system's behavior, at least to the first approximation.

Moreover, although the mean-field solutions for the gravitational potential with no interactions are known in terms of hypergeometric functions, one can, in principle, introduce some approximations also for the gravitational potential. Let us consider a test particle in the star's outermost layer; then its gravitational potential energy is proportional to the product of its masses by the inverse of the distance to the center of the star that concentrates the largest part of the mass. For a spherically symmetric distribution, the first approximation is such that the mass is proportional to the central density times the volume of the sphere. Then, the gravitational potential would be proportional to the distance squared, i.e. a *harmonic oscillator potential*. This approximation is made precise, for instance, in Refs. 39 and 40.

First, let us recall that the mass M_T within a spherically symmetric BS, is given by

$$M_T(r) = 4\pi m_\phi N \int_0^r \rho(r') {r'}^2 dr'. \tag{6}$$

where $0 \leq r \leq R$. The radius R will be considered as the radius of the BS if most of its mass is contained in a region bounded by that radius (often called the R_{99}

radius[28]). N is the corresponding number of particles and $\rho(r)$ is the one particle probability density, which admits a series expansion when most of the matter of the BS is close enough to $r = 0$, i.e.

$$\rho(r) = \rho(r=0) + \sum_{n=1}^{\infty} \frac{\rho^{(n)}(r=0) r^n}{n!}, \tag{7}$$

where $m_\phi N \rho(r=0)$ is the central mass density of the BS, and $\rho^{(n)}$ its corresponding derivatives. However, the density $\rho(r)$ tends to infinity as $r \to \infty$ which is a nonphysical behavior since the density $\rho(r)$ must tend to zero for large r. In order to avoid this unphysical scenario, it is assumed that

$$\left| \frac{\rho^{(n)}(r=0)}{\rho(r=0)} \right| \approx \frac{1}{R^n}. \tag{8}$$

Now, for the motion of a test particle that goes through the BS along the collinear diameter with the z-axis, the density peak is at the center of the BS, yielding

$$M_T(r) = \frac{4}{3}\pi m_\phi N \rho(r=0) r^3 \left[1 + 3\sum_{n=2}^{\infty} \frac{1}{(n+3)n!} \left(\frac{r}{R}\right)^n \right]. \tag{9}$$

The sum in Eq. (9) must be convergent but small by hypothesis. Indeed, in Ref. 39, it is shown that $\forall\delta$ there is an index of some term of the sum such that, for all the larger terms, the sum is bounded by δ. For δ small enough but not negligible, the remaining terms of the sum can be considered as perturbations. Therefore, the leading term of the potential is reduced to a harmonic potential with an *effective gravitational frequency* given by

$$\omega_g = \sqrt{\alpha \pi G m_\phi N \rho(r=0)}, \tag{10}$$

where G is the Newtonian constant of gravitation. Here also $\alpha = 4/3 + 4\delta$. If $\delta = 10^{-1}$, corresponding to be less than 10% of the total mass, the factor 52/30 of Ref. 39 is recovered. Briefly, we can define an *effective gravitational frequency*, so that the gravitational potential can be interpreted at first order, as a trapping harmonic-like potential, as occurs in the usual BEC's formalism.

3. Boson Star Structural Analysis

According to the conditions obtained in the previous section, the corresponding analysis for the properties related to the BS can be summarized as follows, the trapping potential is given by

$$V_g \approx \frac{1}{2} m_\phi \omega_g^2 r^2, \tag{11}$$

and the approximation for the total mass

$$M_T \approx 4\pi N m_\phi \int_0^R \rho(r=0) r^2 dr. \tag{12}$$

On the other hand, if we further assume that most of the particles are inside the condensate, that is, in the $\mathbf{p} = 0$ state then, this implies that the number of particles in the excited states is negligible for temperatures $T < T_c$, where T_c is the condensation temperature. The contributions of the particles in the excited states could affect the properties of the system. See, for instance, Ref. 39. However, we consider here that almost all the particles lie in the corresponding ground state according to the approximation equation (13), (see below). The contributions of the excited states could be important in the stability analysis and will be studied in future works. Thus, the last assertions can be expressed as follows:

$$N_0 \approx N, \quad \sum_{\mathbf{p}\neq 0} N_{\mathbf{p}} \ll N, \tag{13}$$

being N the total number of particles, $N_{\mathbf{p}}$ the number of particles in the excited states, and N_0 the number of particles in the ground state. Keeping terms up to second order in $\hat{a}_0$ and $\hat{a}_0^\dagger$, i.e. $\langle \hat{a}_0^\dagger \hat{a}_0 \rangle = \langle N \rangle$, we are able to obtain the ground state energy (E_0) associated with our BS

$$E_0 = -\frac{\hbar^2}{2m_\phi}\langle 0|\nabla^2|0\rangle N + \frac{1}{2}\langle 0,0|U_0|0,0\rangle N^2 + \langle 0|V_g|0\rangle N. \tag{14}$$

Thus, we have for instance the kinetic energy

$$\langle 0|\nabla^2|0\rangle = \int_0^\infty \int_\Omega \Psi_0^*(r)\nabla^2\Psi_0(r) r^2 \sin\theta dr d\theta d\phi, \tag{15}$$

and for each of the energy contributions to the ground state equation (14).

At this point, we introduce some well-known *ansätze* for the single-particle wave function $\Psi_0(r)$, which is summarized in Table 1. There are several ansätzes used in the literature. As the wave functions for BS usually do not have compact support,

Table 1. Ansätze table for the wave function of a single particle and its corresponding parameters $A, \kappa, \epsilon_1, \epsilon_3, \epsilon_3$. The ansätzes presented here are known in the literature as Gaussian (G), exponential (E), linear–exponential (LE) and compact (C), respectively.

	G	E	LE	C
$\Psi_0(r)$	$(\frac{\beta^2}{\pi})^{3/4} e^{-\beta^2 r^2/2}$	$(\frac{\beta^2}{\pi^{2/3}})^{3/4} e^{-\beta r}$	$(\frac{\beta^2}{7^{2/3}\pi^2})^{3/4}(1+r\beta)e^{-\beta r}$	$\sqrt{\frac{4\pi\beta^3}{(2\pi^2-15)}}\cos^2(\frac{\pi\beta r}{2})$
A	$\pi^{-3/2}$	π^{-1}	$\frac{1}{7\pi^3}$	$\frac{4\pi}{(2\pi^2-15)}$
κ	2.8	4.2	5.4	1
ϵ_1	$\frac{3}{4}$	$\frac{1}{2}$	$\frac{3}{14\pi^2}$	$\frac{(4\pi^2-6)\pi^2}{(24\pi^2-180)}$
ϵ_2	$\frac{3}{4}$	$\frac{1}{2}$	$\frac{81}{28\pi^2}$	$\frac{3(2\pi^4-5\pi^2+315)}{10(2\pi^3-15\pi)}$
ϵ_3	$\frac{1}{2(2\pi)^{3/2}}$	$\frac{1}{16\pi}$	$\frac{437}{25088\pi^5}$	$\frac{35(24\pi^3-205\pi)}{288(2\pi^2-15)}$

three noncompact *ansätze* are proposed, see, for instance, Ref. 28 and references therein. However, in the Thomas–Fermi approximation, the BS can have a fixed radius, which may have some advantages,[29] for which we also propose a compact ansatz. Sometimes, these functions contain adjustable parameters to compare with the numerical solutions.[28] As we see in Table 1, each of our proposed wave functions has a single parameter β with units of the inverse of length. In principle, this parameter can be different in each case, but due to the approximation of the harmonic potential, we will assume that it fulfills $\beta = \sqrt{\frac{m_\phi \omega_g}{\hbar}}$. Moreover, the interpretation of β is related to the BS's radius, as discussed below.

On the other hand, the probability density is the square of the wave function, $\rho(r) = |\Psi_0(r)|^2$. From this definition, we can obtain the corresponding central density evaluating at $r = 0$, i.e.

$$\rho(r = 0) = A\beta^3, \tag{16}$$

where A is a numerical factor that depends on whether it is Gaussian (G), Exponential (E), Linear–Exponential (LE), or Compact (C) anzatz, according to Table 1. Note that the central density in each case is expressed in terms of the parameter β. We can substitute these central densities in the expression for the *effective gravitational frequency* equation (10) that gives us

$$\omega_g = \sqrt{\alpha\pi A G m_\phi N \beta^3}, \tag{17}$$

also in terms of the inverse length β.

Let us realize that β depends on the *effective gravitational frequency*, and this, in turn, depends on the central density, i.e. by using Eq. (16) we obtain

$$\begin{aligned} \rho(r = 0) &= A\beta^3 = A\left(\frac{m_\phi \omega_g}{\hbar}\right)^{3/2} \\ &= A\left(\frac{m_\phi}{\hbar}\right)^{3/2} (\alpha G\pi m_\phi N \rho(r = 0))^{3/4}. \end{aligned} \tag{18}$$

Therefore, if the *ansätze* given in Table 1 and the potential equation (11) be compatible, then the following expression for the central density of the BS must be consistent:

$$\rho(r = 0) = \left(\alpha G\pi \frac{m_\phi^3}{\hbar^2}\right)^3 A^4 N^3. \tag{19}$$

Note that for all practical purposes, the *effective gravitational frequency* for each ansatz has the same functional form, qualitatively speaking, and consequently, the functional form of the central density for each ansatz has this shape.

On the other hand, with the expression for β in terms of the *effective gravitational frequency*, if we substitute ω_g from Eq. (10), and the central density from Eq. (19), we obtain an order of magnitude for the inverse length parameter β for each *ansätze*, namely

$$\beta = \frac{\alpha\pi GAN}{\hbar^2} m_\phi^3. \tag{20}$$

Almost all the considered *ansätzes* do not have a compact support, i.e. they lack a defined surface that contains them since they are infinitely extended objects. Thus, parameter β can be related to the standard size of the BS, the so-called R_{99}, an effective radius that defines a spherical surface within which 99% of the star's mass is enclosed. Both are related through a fixed value $\kappa = \beta R_{99}$ which depends on the chosen ansatz,[28] as shown in Table 1. In the case of the compact ansatz, the radius is exactly the inverse of β parameter, i.e. $R = \beta^{-1}$. To make a unified treatment of the ansätze, in this work, we will use R for the radius and in the compact case $\kappa = 1$.

Let us calculate the corresponding ground state energy of the BS, by substituting each *ansätze* into the ground state energy equation (14). Then, we obtain the following expression for the ground state energy E_0:

$$E_0 = \epsilon_1 \frac{\hbar^2\beta^2 N}{m_\phi} + \epsilon_2 \frac{m_\phi \omega_g^2 N}{\beta^2} + \epsilon_3 U_0 \beta^3 N^2, \tag{21}$$

where the numerical coefficients ϵ_i differ for each ansätze and are also shown in Table 1. Note that the radial integral in Eq. (14) cannot be performed up to infinity for the compact ansatz case, since one need to ask that the function vanishes for radii greater than β^{-1}. In other circumstances, it is known that this can lead to some difficulties.[28] However, in this case, it is enough to integrate up to β^{-1} to obtain the numerical coefficients above.

In order to obtain the thermodynamic quantities, it will be necessary to replace β and ω_g as functions of the volume. Since we are considering a spherically symmetric BS, the available volume in the ideal case, i.e. when the interactions among the constituents within the BS are neglected, is as follows $V_{\mathrm{BS}} = 4\pi\kappa^3\beta^{-3}/3$ then, the ground state energy becomes:

$$\begin{aligned} E_0 = {} & \epsilon_1 \frac{\hbar^2 N}{m_\phi} \left(\frac{4\pi\kappa^3}{3}\right)^{2/3} V_{\mathrm{BS}}^{-2/3} + \epsilon_2 \pi A\alpha G m_\phi^2 N^2 \left(\frac{4\pi\kappa^3}{3}\right)^{1/3} V_{\mathrm{BS}}^{-1/3} \\ & + \epsilon_3 U_0 N^2 \left(\frac{4\pi\kappa^3}{3}\right) V_{\mathrm{BS}}^{-1}. \end{aligned} \tag{22}$$

From the N-body ground state energy equation (22), we therefore calculate the ground state pressure $P_0 = -\frac{\partial E_0}{\partial V_{\mathrm{BS}}}$ for each *ansätze*, with the result

$$P_0 = \left(\frac{3}{4\pi\kappa^3}\right) \left[\frac{2}{3}\epsilon_1 \frac{\hbar^2 N}{m_\phi}\beta^5 + \frac{1}{3}\epsilon_2 \pi A\alpha G m_\phi^2 N^2 \beta^4 + \epsilon_3 U_0 N^2 \beta^6\right]. \tag{23}$$

After rearranging the terms by identifying the scale β from Eq. (20), we can obtain the following two terms, which are of the same order in the length scale that those usually found, but with different coefficients

$$P_0 = \left(\frac{3}{4\pi\kappa^3}\right)\left[\frac{(2\epsilon_1+\epsilon_2)}{3}\frac{\hbar^2 N}{m_\phi}\beta^5 + \epsilon_3 U_0 N^2 \beta^6\right]. \tag{24}$$

If we assume that the pressure and gravity allows the BS to remain in equilibrium, then the following constraint to the number of particles is reached N_e

$$N_e = \frac{(2\epsilon_1+\epsilon_2)}{12\pi\kappa\epsilon_3}\frac{R}{|a|}, \tag{25}$$

where N is several orders of magnitude greater than κ. The scattering length, whose value can also be negative, should be only constrained from the particle physics model, this is, from Eq. (25). For a given value of a, we should know the region where systems are not allowed to exist due to equilibrium. If $N < N_e$ gravity overcomes the pressure and we could have an implosion of the system. If $N > N_e$, then pressure overcomes gravity, and apparently, the system becomes unstable. N_e allows us to find the systems that are in equilibrium, and stability conditions will be studied in a future work, where additional properties as rotation or more general self-interacting potentials within the BS can also be included.

Within the Thomas–Fermi approximation, we compute the mass and pressure of a boson star, which are functions of the individual boson mass particle, m_ϕ, the system's number of particles, N, and the scattering length, a. The radius of the boson star is computed assuming gravity-pressure stability, Eq. (25). Therefore, in this approach, we conclude that a nonrelativistic boson star may exist in a wide range of masses, radii, and pressure depending on the value of the parameters of the model and Fig. 1 shows this conclusion in N–m_ϕ space parameter. For each panel of Fig. 1, we show extreme values that nonrelativistic boson star may have, for instance, for the radius we assume objects from Sun-size to DM halo size, 10^3 kpc,[41] and anything in the range may be possible with the right combination of parameters. The extreme values for the mass panel were Sun-mass and dark matter halo, $10^{12}\,\mathrm{M}_\odot$.[41] The extreme values for the pressure panel are as slow as 1 atmosphere to 10^{31} Pa for a neutron star,[42] the last is a pure reference since we clarify that this approach is only valid in the nonrelativistic limit, see Table 2 for numerical values of previous examples.

Alternatively to this scenario, a phenomenological stability condition has been proposed in Ref. 43 for a trapped laboratory BEC. For a system with attractive interactions, i.e. $a < 0$, there is not enough kinetic energy to stabilize the BEC, and it is expected to collapse for a sufficiently large number of particles. A BEC can avoid collapse only as long as the number of atoms is less than a critical value given by

$$N_c = \gamma^2 \frac{R}{|a|}, \tag{26}$$

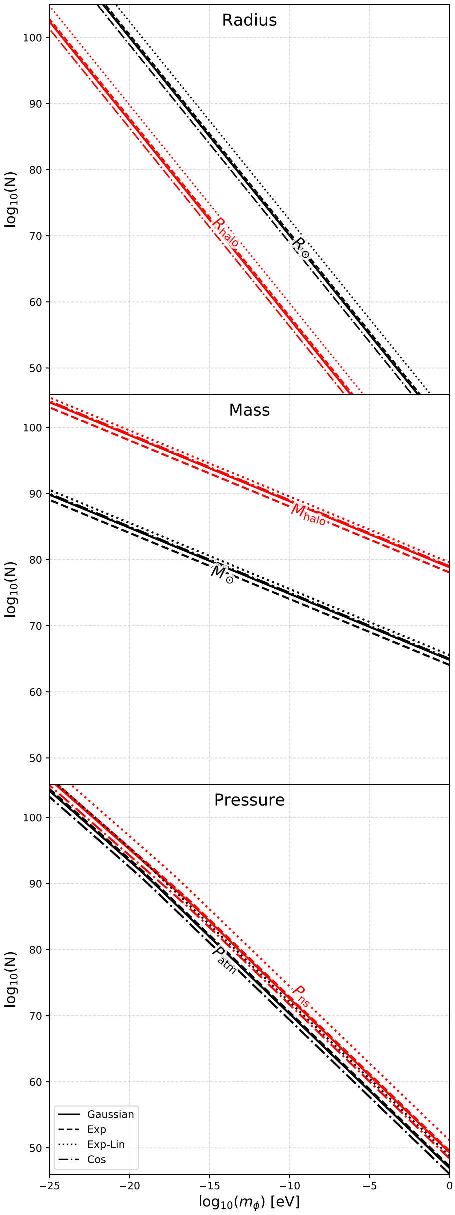

Fig. 1. (Color online) Plots of the Radii (top panel), Mass (middle panel) and Pressure (Bottom panel) for a Boson star. Red (and black) lines represent extreme high (low) values for each case, for the radii panel are a dark matter halo characteristic radius of 10^3 kpc,[41] $R_{\rm halo}$ (in red), and the Solar radius, $R_\odot$ in black. In the case of the mass, we show contour lines for a solar mass, $M_\odot$ (in black), and a typical mass for dark matter halos $10^{12}\ M_\odot$,[41] $M_{\rm halo}$ (in red). For the pressure, we show lines that represent the pressure of a neutron star $\mathcal{O}(10^{31})$ Pa,[42] $P_{\rm ns}$ (in red), and 1 atmosphere of pressure, $P_{\rm atm}$ (in black). Different line-styles represent different approximations, straight, dashed, dotted, dash-dotted represent the Gaussian, linear–exponential, exponential and cosine, respectively.

where the parameter γ^2 is the so-called stability coefficient[a] and R the size of the system. The stability coefficient depends on the properties of the trapping potential; see Ref. 43 for details. Thus, according to our model, the stability coefficient is in each case

$$\gamma^2 = \frac{(2\epsilon_1 + \epsilon_2)}{12\pi\kappa\epsilon_3}. \tag{27}$$

By inserting Eq. (26) into Eq. (24) with $a > 0$, the pressure simplifies as follows:

$$P_e = \frac{(2\epsilon_1 + \epsilon_2)^2}{24\pi^2\epsilon_3}\frac{\kappa\hbar^2}{am_\phi R_e^4}, \tag{28}$$

where the subindex e means that is evaluated at gravitational equilibrium, $N = N_e$.

To compute the equilibrium number of particles of the system, N_e, we still need to solve Eq. (26) because R is function of the number of particles, N. Using $R = \kappa/\beta$ and Eq. (20) we obtain

$$N_e = \left(\frac{(2\epsilon_1 + \epsilon_2)\hbar^2}{12\pi^2 A\epsilon_3 aGm_\phi^3\alpha}\right)^{1/2}, \tag{29}$$

as a consequence, the only free parameter is the scattering length, a.

Table 2. Values to form a Sun-like or galaxy cluster halo system for different ansätzes the Gaussian, the exponential, the linear exponential (Lin–Exp), and the compact. m_ϕ is the mass of the particle, N is the number of particles, a is the scattering length, and P is the pressure of the system.

		Gaussian	Exponential	Lin–Exp	Compact
Sun-like	m_ϕ [eV]	1.29×10^{-12}	5.15×10^{-12}	8.51×10^{-12}	2.92×10^{-13}
	N	5.23×10^{76}	2.19×10^{75}	4.31×10^{76}	3.44×10^{77}
	a [m]	9.6×10^{-69}	3.96×10^{-67}	6.35×10^{-66}	2.47×10^{-63}
	P [Pa]	3.03×10^{12}	1.17×10^{11}	2.66×10^{12}	2.75×10^{20}
Dwarf halo	m_ϕ [eV]	1.35×10^{-22}	5.37×10^{-22}	8.87×10^{-22}	3.04×10^{-23}
	N	5.02×10^{94}	2.1×10^{93}	4.14×10^{94}	3.3×10^{95}
	a [m]	9.21×10^{-75}	3.8×10^{-73}	6.09×10^{-72}	2.37×10^{-69}
	P [Pa]	4.22×10^{-20}	1.62×10^{-21}	3.7×10^{-20}	3.83×10^{-12}
Cluster halo	m_ϕ [eV]	2.01×10^{-26}	8.02×10^{-26}	1.33×10^{-25}	4.54×10^{-27}
	N	3.36×10^{104}	1.41×10^{103}	2.77×10^{104}	2.21×10^{105}
	a [m]	6.17×10^{-83}	2.54×10^{-81}	4.08×10^{-80}	1.59×10^{-77}
	P [Pa]	1.05×10^{-14}	4.03×10^{-16}	9.17×10^{-15}	9.5×10^{-7}

[a]In usual laboratory Bose–Einstein condensates, this parameter is a positive dimensionless constant. Also, its value depends on some properties of the trap and is related to the system's stability, see, for instance, Ref. 43.

4. Numerical Analysis

In Fig. 2, black lines represent systems in equilibrium, in all the figures we have taken two extreme examples. The dashed line is defined using $a_\odot$ is for a system of with a mass ($M_e = M_\odot$) and radius ($R_e = R_\odot$) as the Sun. Dash-dotted lines are for a system defined with the value of a_h to represent a system of the size and mass of a typical cluster of dark matter halo, this is $M_e = 10^{14}$ $\mathrm{M}_\odot$ and $R_e = 10^3\,\mathrm{kpc}$.[44]

Interestingly, with only one parameter, this approach can predict BECs of very different scales and may consider a fine-tuning problem to differentiate the value of the scattering length a. The larger value for a is given to describe Sun-like systems, and the compact hypothesis gives its largest values, which is of the order of 10^{-63} m. For each of the hypotheses taken, the values of a are different depending on if one wants to describe a Sun-like, dwarf DM halo, or a cluster DM halo system, see Appendix A in which we show all the cases. It is interesting that independent of the hypothesis taken, the value of $a_{\rm sun}$ is 14 orders of magnitude bigger than the case for the cluster dark matter halo, a_h.

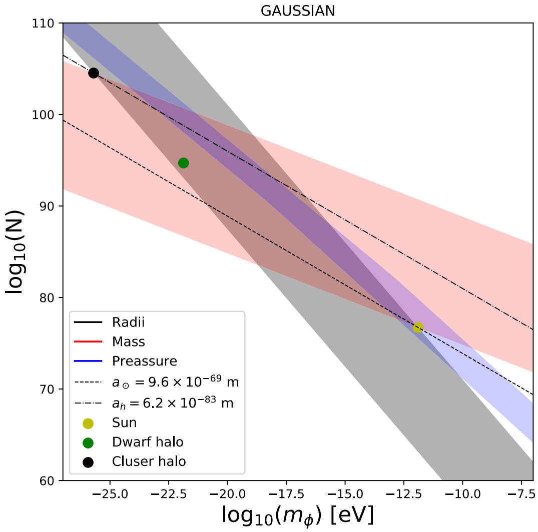

Fig. 2. (Color online) Contour plots where shaded regions may represent realistic astrophysical systems. The black region represents a system of the size between $R_\odot$ and a typical dark matter halo in a galaxy cluster 10^3 kpc. The red region represents a system between a solar mass ($M_\odot$) and the mass of a dark matter halo in a galaxy cluster (10^{14} $M_\odot$). The blue region represents systems between 1 atmosphere and inner crust neutron star (10^{31} Pa) pressure. Black lines are system in equilibrium given by Eq. (29), dashed (dash-dotted) line is taken a for a Sun-like (dark matter halo kind) system. Yellow, green, and black dots represent the Sun, dwarf DM halo, and galaxy cluster dark matter halo system, respectively.

Additionally, by using Eqs. (22), (24), and (29) we can find a relation between the pressure, P, and the energy density, ϱ_ϵ, to compute the EoS of the type $P = \omega\varrho_\epsilon$. Thus, we obtain

$$\omega = \frac{1 + \left(\dfrac{N}{N_e}\right)^2}{\dfrac{3(\epsilon_1 + \epsilon_2)}{2\epsilon_1 + \epsilon_2} + \left(\dfrac{N}{N_e}\right)^2}. \tag{30}$$

When $N \gg N_e$ then $\omega \sim 1$ this may describe systems with properties known as a *stiff matter*, see for instance Ref. 45, and references therein for details on this kind of schemes. For systems in equilibrium, Eq. (30) simplifies

$$\omega = \frac{1 + \dfrac{2\epsilon_1}{\epsilon_2}}{2\left(1 + \dfrac{5\epsilon_1}{4\epsilon_2}\right)}, \tag{31}$$

then, for the Gaussian and exponential case we have $\epsilon_1/\epsilon_2 = 1$, this is, $\omega = 2/3$ describing an ideal system. For the linear-exponential case $\epsilon_1/\epsilon_2 = 2/27$ and $\omega = 31/59 \approx 0.53$ and the compact case $\omega \approx 0.63$. Note that for systems in equilibrium, the corresponding EoS does not depend on any of the variables for the different *ansätzes*, and this also coincides with the fact that the scattering length, a, is quite small, in the sense that reflects the nature of a quasi-ideal nonrelativistic gas.

Despite the value of ω in the EoS, the fact that the system's pressure, Eq. (28), depends on a makes possible to form halos in equilibrium of a galaxy cluster size with very slight pressure. This characteristic resembles the properties of dark matter. For instance, to form a system with the properties of a dark matter halo that surround a galaxy cluster in the Gaussian case, we would need a scattering length of order $a_h = 6.17 \times 10^{-83}\,\mathrm{m}$, a boson particle of mass $m_\phi = 2.01 \times 10^{-26}\,\mathrm{eV}$, and $N = 3.36 \times 10^{104}$ particles, and the BEC would have a pressure of $1.05 \times 10^{-14}\,\mathrm{Pa}$, it is the extension of the condensate that makes it plausible to the existence of this kind of systems.

Moreover, according to our approach in order to form system with mass and size of the Sun, for instance, we would need $N = 5.23 \times 10^{76}$ particles of mass $m_\phi = 1.29 \times 10^{-12}\,\mathrm{eV}$ with an scattering length $a_{\mathrm{sun}} = 9.6 \times 10^{-69}\,\mathrm{m}$. This Sun-like systems would have a pressure of $P_e = 3.03 \times 10^{12}\,\mathrm{Pa}$. The shadowed areas in Fig. 2 represent systems with reasonable radii, mass, and pressure for an astrophysical system. It is also interesting that systems with very high pressure can be found in this scheme, for instance, systems with $N = 9.3 \times 10^{91}$ particles with mass $m_\phi = 8.0 \times 10^{-18}\,\mathrm{eV}$ and scattering length $a = 9.6 \times 10^{-84}\,\mathrm{m}$ would have a pressure of $P = 1.5 \times 10^{31}\,\mathrm{Pa}$, which is of the order of the inner crust pressure of a neutron star.[42] However, the approach we are proposing may not be valid to such high pressures, and relativistic corrections may have to be taken into account. Note also that when quantum effects are taken into account, i.e. for $T \to 0$, in which Pauli's principle discriminate between bosons and fermions, the pressure tends to be zero at zero temperature in the case of bosons, which is not the case for fermions.[46] Conversely, the EoS $P = \varrho_\epsilon$ in which the pressure is proportional to the energy density ρ_ϵ is known in the literature as the so-called *stiff matter*, see, for instance, Refs. 45 and 47 and references therein. We must mention that we are able to obtain a *stiff matter*-like EoS here at $T \to 0$, even when our BS lies in the nonrelativistic and low-density regime. The topics mentioned above deserve more in-depth analysis and will be presented elsewhere.[48]

Finally, several previous works[49–55] have put constraints on the mass of scalar fields by using galaxy rotation curves of dwarf galaxies since it is believed that the DM halo dominates the kinematic of this kind of systems. A typical size of the DM halo for a dwarf galaxy is of the order 22 kpc and has a total mass of the order of 10^8 $M_\odot$.[56,57] To form systems with these features, we would need a boson particle with a mass of $m_\phi = 1.35 \times 10^{-22}\,\mathrm{eV}$ and a scattering length $a = 9.21 \times 10^{-75}\,\mathrm{m}$, this is consistent within an order of magnitude with the previous results, and it is consistent with the DM as dust hypothesis since the corresponding pressure is very small $P = 4.22 \times 10^{-20}\,\mathrm{Pa}$.

We must mention that a relevant difference among the different *ansätzes* used in this work is the order of magnitude associated with the pressure of the systems within the compact scenario, i.e. the compact *ansatz*, in which case is up to 9 orders of magnitude bigger than the exponential *ansatz* for the cluster halo and the Sun-like examples. It seems to be that this discrepancy in the pressure between the compact and the non-compact *ansätze* relies on the choice for the corresponding BS's radius $R = \beta^{-1}$, i.e. $\kappa = 1$ for the compact *ansatz*, see Eq. (28). In other words, the results obtained in this work agree with the previous results reported in Ref. 28, which suggests that the case of the compact *ansatz* deserves deeper study and should be handled carefully.

5. Conclusions

We have analyzed a collection of nonrelativistic gravitational bounded generic bosons forming a Bose–Einstein condensate starting from the single-particle properties. We have also shown that the system admits equilibrium configurations for various scenarios that can be interpreted as BS and perhaps dark matter. By using four *ansätze*, the Gaussian, Exponential, the Linear exponential (noncompact ansatz), and the Cosine (compact ansatz), we can prove that they predict almost the same structural configuration for the BS, qualitatively speaking. Additionally, we have shown that different values of the corresponding scattering length, together with some specific values of the corresponding number of particles, lead to several sizes of BS in gravitational equilibrium. With our model, we are able to describe the structural equilibrium configuration from compact objects (i.e. the size of the sun, for instance) to gigantic configurations comparable to a galaxy cluster dark matter halos. In other words, our model predicts several configurations that may be stable and can form systems in gravitational equilibrium in a wide range of sizes. Note that we can also extract significant properties associated with the BS thermodynamics. For instance, concerning the EoS, we have calculated the corresponding ground state energy for each ansatz, from which we can obtain the corresponding internal energy and, consequently, the corresponding pressure. We can define two apparent limits according to the definition of the pressure $P = -\partial E_0/\partial V_{\rm BS}$. The first one corresponds to the ideal case, i.e. $U_0 = 0$, and the second one at zero temperature. For the first case, i.e. the ideal system, we obtain that the EoS is given by $P = \frac{2}{3}\frac{E_0}{V_{\rm BS}} = \frac{2}{3}\varrho_\epsilon$, being ϱ_ϵ the energy density. Conversely, when $T \rightarrow 0$ and when the contributions of the interactions dominate, the EoS can be expressed approximately as $P = \frac{E_0}{V_{\rm BS}} = \varrho_\epsilon$. Note that $P = \frac{2}{3}\varrho_\epsilon$ is the standard EoS for an ideal nonrelativistic bosonic system (it can also be proved that this is the EoS for a system of ideal nonrelativistic fermions in the classical regime[46]). Additionally, when interactions are neglected, i.e. $U_0 = 0$, the pressure also tends to zero when $T \rightarrow 0$, i.e. the system behaves as dust, when the quantum nature of the bosonic particles in the condensed phase is taken into account.[46] Moreover, the EoS $P = \varrho_\epsilon$ in which the pressure is proportional to the energy density ϱ_ϵ is known in the literature as the so-called *stiff matter*.[b] We must mention that we are able to obtain apparently a *stiff matter* EoS when $T \rightarrow 0$, even when our BS lies in the nonrelativistic and low-density regime.

The topics mentioned above deserve more in-depth analysis and will be presented elsewhere.[48] Finally, this work must be extended to rotating systems in order to analyze the gravitational equilibrium and the corresponding stability. Moreover, more general interactions within the system could also be relevant for compact objects, i.e. three-body interactions. Consequently, the analysis of logarithmic-like potentials, which describe multi-body interactions within the system, can help analyze the structural configuration of the nonrelativistic and the relativistic limits associated with BSs in the line of Ref. 58. A further issue related to the BS configurations described in this work is that they can be useful, also to estimate the amount of DM in the solar system. Finally, with an exhaustive analysis of these results, it might be possible to explore the relationship of BS and DM as BEC of generic bosons in the universe.

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[b] Note that we can also express the pressure when $T \rightarrow 0$ as a function of the density of particles ρ when interaction dominates. In such a case, we obtain $P = \omega_d \rho^2$ with $\omega_d = \epsilon_1 \frac{\hbar^2}{m_\phi}(\frac{4\pi\kappa^3}{3})$.

Appendix A. Plots

Here we present the contour plots for the exponential, linear–exponential, and compact cases.

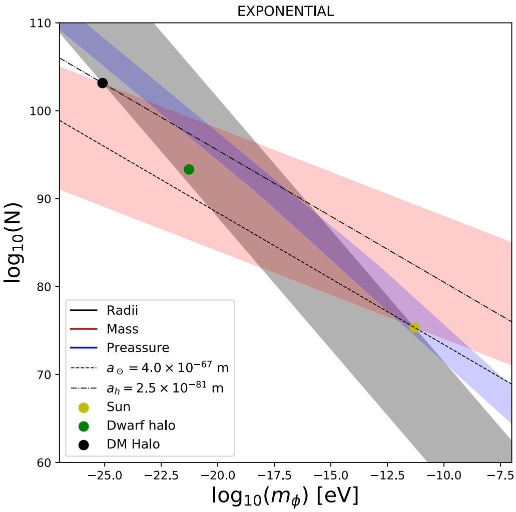

Fig. A.1. (Color online) Contour plots for the exponential ansatz. Colors and styles are the same as in Fig. 2.

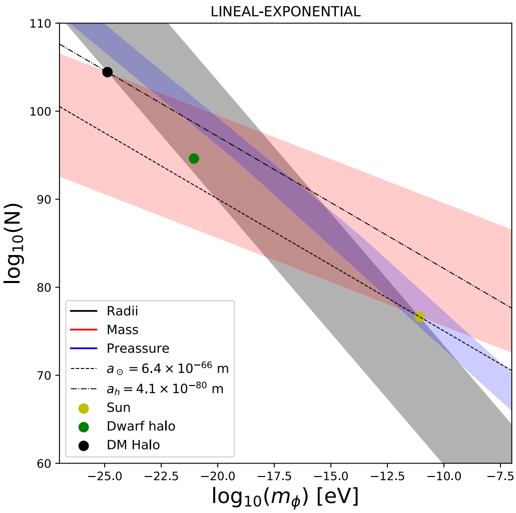

Fig. A.2. (Color online) Contour plots for the linear–exponential ansatz. Colors and styles are the same as in Fig. 2.

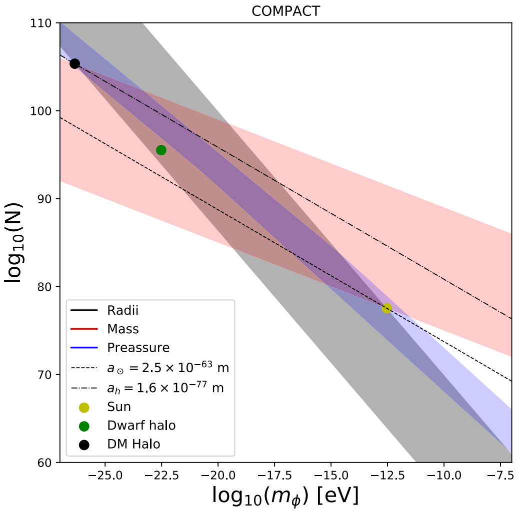

Fig. A.3. (Color online) Contour plots for the compact ansatz. Colors and styles are the same as in Fig. 2.

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